

Transverse Kinetic Stability*

Steven M. Lund
Lawrence Livermore National Laboratory (LLNL)

Steven M. Lund and John J. Barnard
“Beam Physics with Intense Space-Charge”

US Particle Accelerator School

University of Maryland, held at Annapolis, MD

16-27 June, 2008
(Version 20080625)

* Research supported by the US Dept. of Energy at LLNL and LBNL under contract Nos. DE-AC52-07NA27344 and DE-AC02-05CH11231.

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Overview: Machine Operating Points	SM Lund, USPAS, June 2008	Transverse Kinetic Stability 2
Linearized Vlasov Equation		
Collective Modes on a KV Equilibrium Beam		
Global Conservation Constraints		
Kinetic Stability Theorem		
rms Emittance Growth and Nonlinear Fields		
rms Emittance Growth and Nonlinear Space-Charge Fields		
Uniform Density Beams and Extreme Energy States		
Collective Relaxation and rms Emittance Growth		
Halo Induced Mechanism of Higher Order Instability		
Phase Mixing and Landau Damping in Beams		
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Detailed Outline - 3

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10) Collective Relaxation and rms Emittance Growth

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Halo Model for an Elliptical Beam

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12) Phase Mixing and Landau Damping in Beams

(to be added, future editions)

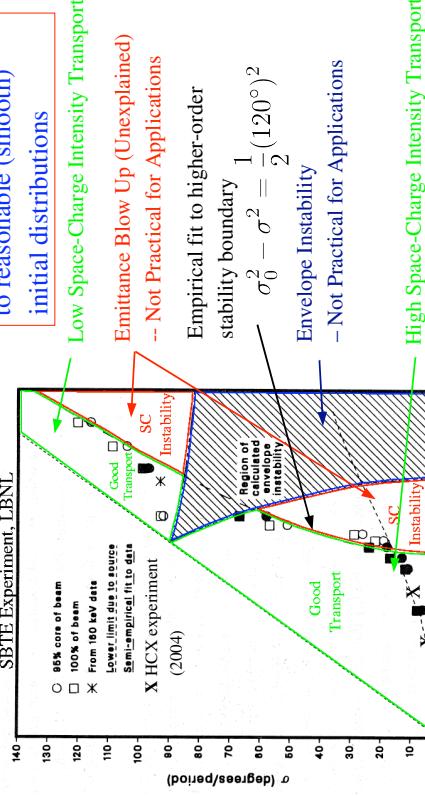
Contact Information

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Transport limits in periodic (FODO) quadrupole lattices that result from higher order processes have been measured in the SBTE experiment. These results had only limited theoretical understanding over 20+ years

Experimental limits on beam stability in terms of σ and σ_0
SBTE Experiment, LBNL
Limits defined with respect to reasonable (smooth) initial distributions



[M.G. Tiefenbach, Ph.D Thesis, UC Berkeley (1986)]
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S1: Overview: Machine Operating Points

Good transport of a single component beam with intense space-charge described by a Vlasov-Poisson type model requires:

1. Lowest Order:

Stable single-particle centroid: $\sigma_0 < 180^\circ$ see: Transverse Particle Eqns, Transverse Centroid and Env.

2. Next Order:

Stable rms envelope: $\sigma_0, \sigma/\sigma_0$ both outside see: Transverse Centroid and Envelope Descriptions of envelope bands

3. Higher Order:

“Stable” Vlasov description: To be covered these lectures

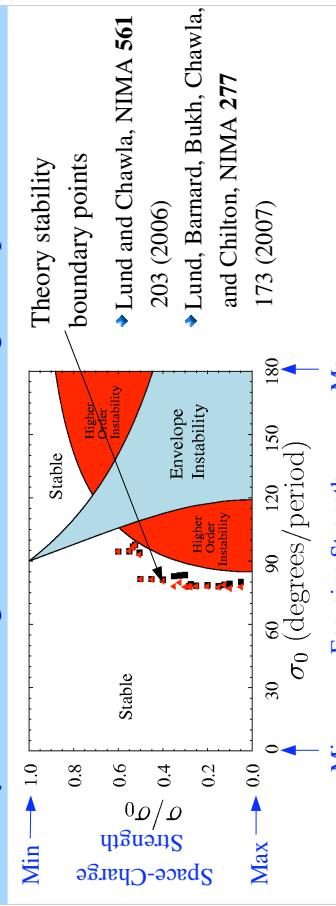
Transport of a relatively smooth initial beam distribution can fail or become “unstable” within the Vlasov model for several reasons:

- Collective modes internal to beam become unstable and grow
 - Large amplitudes can lead to statistical (rms) beam emittance growth
- Excessive halo generated
 - Increased statistical beam emittance and particle losses

• Combined processes above

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Summary of beam stability with intense space-charge in a quadrupole transport lattice: centroid, envelope, and theory boundary based on higher order emittance growth/particle losses



Recent theory analyzes AG transport limits without equilibria

- ◆ Suggests near core, chaotic halo resonances driven by matched beam envelope flutter can drive strong emittance growth and particle losses

◆ Results checked with fully self-consistent simulations

Analogous results (with less “instability”) exist for solenoidal transport

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S2: Overview:

Collective Modes and Transverse Kinetic Stability

In discussion of transverse beam physics we have focused on:

Equilibrium

- Used to estimate balance of space-charge and focusing forces
- KV model for periodic focusing
 - Continuous focusing equilibria for qualitative guide on space-charge effects such as Debye screening and nonlinear equilibrium self-field effects

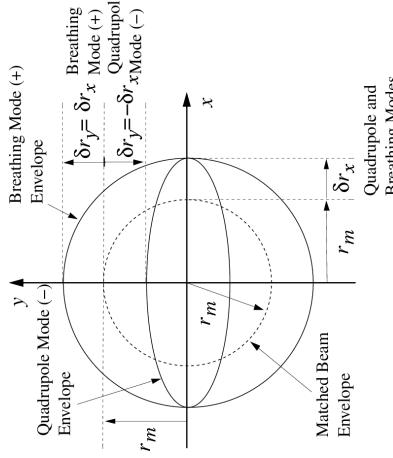
Centroid/Envelope Modes and Stability

- Lowest order collective oscillations of the beam
 - Analyzed assuming fixed internal form of the distribution
- Model only exactly correct for KV equilibrium distribution
 - Should hold in a leading-order sense for a wide variety of real beams
- Predictions of instability regions are well verified by experiment
 - Significantly restricts allowed system parameters for periodic focusing lattices

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Example – Envelope Modes on a Round, Continuously Focused Beam



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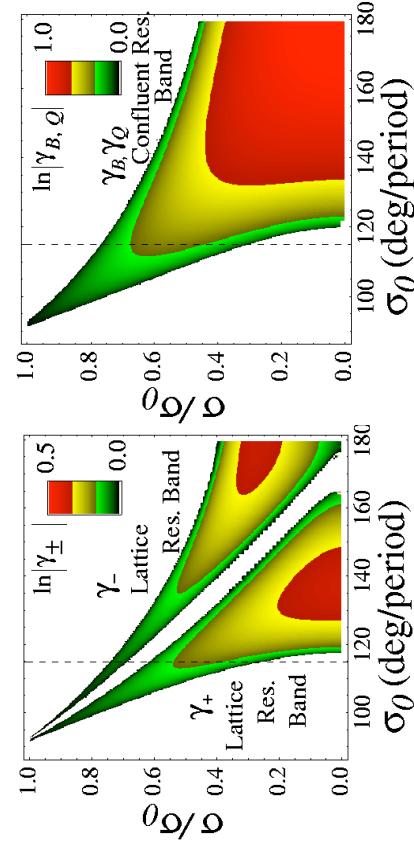
The analog of these modes in a periodic focusing lattice can be destabilized

- Constrains system parameters to avoid band (parametric) regions of instability

Envelope Mode Instability Growth Rates

Solenoid ($\eta = 0.25$)

Quadrupole FODO ($\eta = 0.70$)

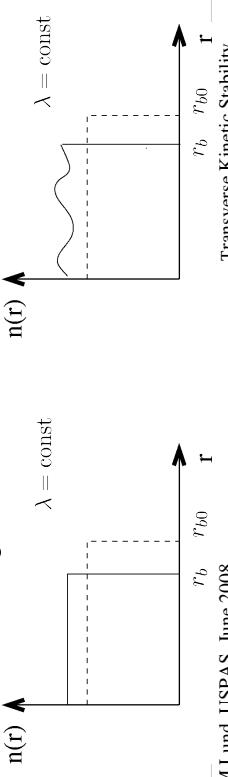


[S.M. Lund and B. Bulth, PRSTAB 024801 (2004)]
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Higher-order Collective (internal) Mode Stability

- Perturbations will generally drive nonlinear space-charge forces
- Evolution of such perturbations can change the beam rms emittance
- Many possible internal modes of oscillation should be possible
 - Frequencies can differ significantly from envelope modes
 - Creates more possibilities for resonant exchanges with a periodic focusing lattice and various beam characteristic responses opening many possibilities for system destabilization

KV Envelope Mode (breathing)



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Plasma physics approach to beam physics:

Resolve:

$$f(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = f_\perp(\{C_i\}) + \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$$

perturbation $f_\perp \gg |\delta f_\perp|$

and carry out equilibrium + stability analysis

Comments:

- Attraction is to parallel the impressive successes of plasma physics
- Gain insight into preferred state of nature
- Beams are born off a source and may not be close to an equilibrium condition
 - Appropriate single particle constants of the motion unknown for periodic focusing lattices other than the KV distribution
 - Intense beam self-fields and finite radial extent vastly complicate equilibrium description and analysis of perturbations

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Review: Transverse Vlasov-Poisson Model: for a coasting, single species beam with electrostatic self-fields propagating in a linear focusing lattice:

$\mathbf{x}_\perp, \mathbf{x}'_\perp$	transverse particle coordinate, angle	$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$	single particle distribution
q, m	charge, mass	$H_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$	single particle Hamiltonian
γ_b, β_b	axial relativistic factors		

VlasovEquation (see J.J. Barnard, [Introductory Lectures](#)):

$$\frac{d}{ds} f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Particle Equations of Motion:

$$\frac{d}{ds} \mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \quad \frac{d}{ds} \mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Hamiltonian (see: S.M. Lund, lectures on [Transverse Equilibrium Distributions](#)):

$$H_\perp = \frac{1}{2} \mathbf{x}'_\perp^2 + \frac{1}{2} \kappa_x(s) x^2 + \frac{1}{2} \kappa_y(s) y^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi$$

Poisson Equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{q}{\epsilon_0} \int d^2 \mathbf{x}'_\perp f_\perp$$

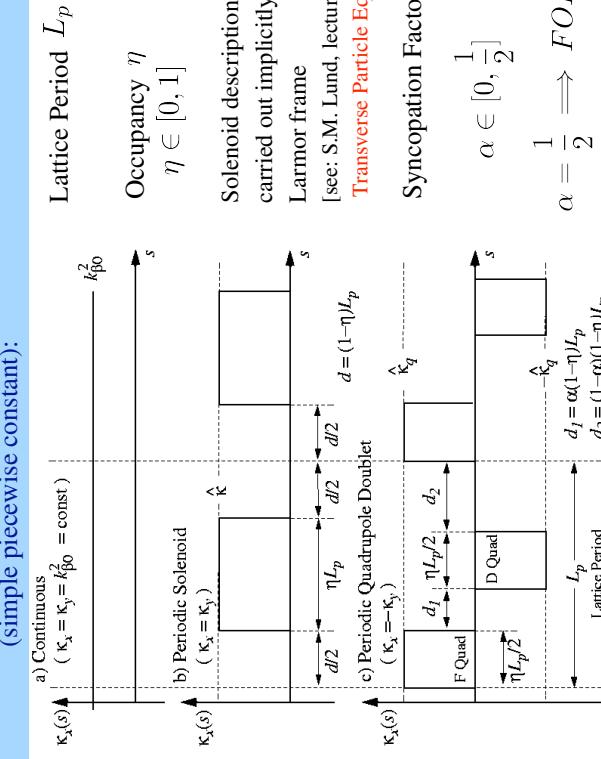
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+ boundary conditions on ϕ

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Review: Focusing lattices, continuous and periodic
(simple piecewise constant):



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Continuous Focusing: $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$H_\perp = \frac{1}{2} \mathbf{x}'_\perp^2 + \frac{1}{2} k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi$$

Solenoidal Focusing (in Larmor frame variables): $\kappa_x = \kappa_y = \kappa(s)$

$$H_\perp = \frac{1}{2} \mathbf{x}'_\perp^2 + \frac{1}{2} \kappa \mathbf{x}_\perp^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi$$

Quadrupole Focusing: $\kappa_x = -\kappa_y = \kappa_q(s)$

$$H_\perp = \frac{1}{2} \mathbf{x}'_\perp^2 + \frac{1}{2} \kappa_q x^2 - \frac{1}{2} \kappa_q y^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi$$

We will concentrate on the continuous focusing model in these lectures

- Kinetic theory is notoriously complicated even in this (simple) case
- By analogy with envelope mode results expect that kinetic theory of periodic focusing systems to have more instabilities
- As in equilibrium analysis the continuous model can give simplified insight on a range of relevant kinetic stability considerations

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S3: Linearized Vlasov Equation

Because of the complexity of kinetic theory, we will limit discussion to a simple continuous focusing model Vlasov-Poisson system for a coasting beam within a round pipe

$$\frac{df_{\perp}}{ds} = \left\{ \frac{\partial}{\partial s} + \mathbf{x}'_{\perp} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} - \left(k_{\beta 0}^2 \mathbf{x}_{\perp} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial \mathbf{x}_{\perp}} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_{\perp}} \right\} f_{\perp}(\mathbf{x}_{\perp}, \mathbf{x}'_{\perp}, s) = 0$$

$$\nabla_{\perp}^2 \phi(\mathbf{x}_{\perp}, s) = -\frac{q}{\epsilon_0} \int d^2 x'_{\perp} f_{\perp}(\mathbf{x}_{\perp}, \mathbf{x}'_{\perp}, s)$$

$$\phi(|\mathbf{x}_{\perp}| = r_p, s) = \text{const}$$

Then expand the distribution and field as:

Comment:

The Poisson equation connects
 f_{\perp} and ϕ so, δf_{\perp} and $\delta \phi$

cannot be independently specified.
 We quantify the connection shortly.

At present, there is *no assumption* that the perturbations are small.

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The equilibrium satisfies:

(see: S.M. Lund, lectures on [Transverse Equilibrium Distributions](#))

$$H_0 = \frac{1}{2} \mathbf{x}'_{\perp} \cdot \mathbf{x}'_{\perp} + \frac{1}{2} k_{\beta 0}^2 \mathbf{x}_{\perp}^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi_0$$

$f_0(H_0) =$ any non-negative function

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi_0}{\partial r} \right) = -\frac{q}{\epsilon_0} \int d^2 x'_{\perp} f_0(H_0)$$

The unperturbed distribution must then satisfy the equilibrium Vlasov equation:

$$\left\{ \frac{\partial}{\partial s} + \mathbf{x}'_{\perp} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} - \left(k_{\beta 0}^2 \mathbf{x}_{\perp} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \mathbf{x}_{\perp}} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_{\perp}} \right\} f_0(H_0) = 0$$

Because the Poisson equation is linear:

$$\nabla_{\perp}^2 \delta \phi(\mathbf{x}_{\perp}, s) = -\frac{q}{\epsilon_0} \int d^2 x'_{\perp} \delta f_{\perp}(\mathbf{x}_{\perp}, \mathbf{x}'_{\perp}, s)$$

$$\delta \phi(|\mathbf{x}_{\perp}| = r_p, s) = \text{const}$$

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Insert the perturbations in Vlasov's equation and expand terms:

$$\begin{aligned} & \left\{ \frac{\partial}{\partial s} + \mathbf{x}'_{\perp} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} - \left(k_{\beta 0}^2 \mathbf{x}_{\perp} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \mathbf{x}_{\perp}} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_{\perp}} \right\} f_0(H_0) \quad \text{equilibrium term} \\ & + \left\{ \frac{\partial}{\partial s} + \mathbf{x}'_{\perp} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} - \left(k_{\beta 0}^2 \mathbf{x}_{\perp} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \mathbf{x}_{\perp}} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_{\perp}} \right\} \delta f_{\perp} \\ & = \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \delta \phi}{\partial \mathbf{x}'_{\perp}} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} f_0(H_0) + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \delta \phi}{\partial \mathbf{x}'_{\perp}} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} \delta f_{\perp} \end{aligned}$$

equilibrium characteristics
of perturbed distribution
perturbed field
nonlinear term
perturbed field
linear correction term
perturbations to be small-amplitude:

$$f_0(H_0) \gg |\delta f_{\perp}|$$

$\phi_0 \gg \delta \phi$ <-- follows automatically from distribution/Poisson Eqn

and neglect the nonlinear terms to obtain the linearized Vlasov-Poisson system:

$$\begin{aligned} & \left\{ \frac{\partial}{\partial s} + \mathbf{x}'_{\perp} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} - \left(k_{\beta 0}^2 \mathbf{x}_{\perp} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \mathbf{x}_{\perp}} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_{\perp}} \right\} \delta f_{\perp}(\mathbf{x}_{\perp}, \mathbf{x}'_{\perp}, s) \\ & = \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \delta \phi(\mathbf{x}_{\perp}, s)}{\partial \mathbf{x}_{\perp}} \cdot \frac{\partial}{\partial \mathbf{x}'_{\perp}} f_0(H_0) \\ & \nabla_{\perp}^2 \delta \phi(\mathbf{x}_{\perp}, s) = -\frac{q}{\epsilon_0} \int d^2 x'_{\perp} \delta f_{\perp}(\mathbf{x}_{\perp}, \mathbf{x}'_{\perp}, s) \quad \delta \phi(|\mathbf{x}_{\perp}| = r_p, s) = \text{const} \end{aligned}$$

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Solution of the Linearized Vlasov Equation, the method of characteristics

The linearized Vlasov equation is a integral-partial differential equation system

- ♦ Highly nontrivial to solve
- ♦ Method of characteristics can be employed to simplify analysis due to the structure of the equation

Note that the equilibrium Vlasov equation is:

$$\left\{ \frac{\partial}{\partial s} + \mathbf{x}'_{\perp} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} - \left(k_{\beta 0}^2 \mathbf{x}_{\perp} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \mathbf{x}_{\perp}} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_{\perp}} \right\} f_0 = 0$$

Interpret:

$$\left\{ \frac{\partial}{\partial s} + \mathbf{x}'_{\perp} \cdot \frac{\partial}{\partial \mathbf{x}_{\perp}} - \left(k_{\beta 0}^2 \mathbf{x}_{\perp} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0}{\partial \mathbf{x}_{\perp}} \right) \cdot \frac{\partial}{\partial \mathbf{x}'_{\perp}} \right\} = \frac{d}{ds|_{\text{eq. orbit}}} \quad f_0 = 0$$

as a total derivative evaluated along an equilibrium particle orbit. This suggests employing the method of characteristics.

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Method of Characteristics:

Equilibrium orbit:

$$\begin{aligned}\frac{d}{d\tilde{s}} \mathbf{x}_\perp(\tilde{s}) &= \mathbf{x}'_\perp(\tilde{s}) \\ \frac{d}{d\tilde{s}} \mathbf{x}'_\perp(\tilde{s}) &= -k_{\beta 0}^2 \mathbf{x}_\perp(\tilde{s}) - \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi_0(\mathbf{x}_\perp(\tilde{s}))}{\partial \mathbf{x}_\perp(\tilde{s})}\end{aligned}$$

“Initial” conditions of characteristic orbit:

$$\begin{aligned}\mathbf{x}_\perp(\tilde{s}=s) &= \mathbf{x}_\perp \\ \mathbf{x}'_\perp(\tilde{s}=s) &= \mathbf{x}'_\perp\end{aligned}$$

Then the linearized Vlasov equation can be expressed as:

$$\frac{d}{d\tilde{s}} \delta f_\perp(\mathbf{x}_\perp(\tilde{s}), \mathbf{x}'_\perp(\tilde{s}), \tilde{s}) = \frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \delta \phi(\mathbf{x}_\perp(\tilde{s}))}{\partial \mathbf{x}_\perp(\tilde{s})} \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} f_0(\mathbf{x}_\perp(\tilde{s}), \mathbf{x}'_\perp(\tilde{s}))$$

This is a total derivative and can be integrated:

- ♦ To analyze instabilities assume growing perturbations that grow in \tilde{s}
- ♦ Neglect initial conditions at $\tilde{s} \rightarrow -\infty$ and integrate

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S4: Collective Modes on a KV Equilibrium Beam

Unfortunately, calculation of normal modes is generally complicated even in continuous focusing. Nevertheless, the normal modes of the KV distribution can be analytically calculated and give insight on the expected collective response of a beam with intense space-charge.

Review: Continuous Focusing KV Equilibrium

$$f_\perp(H_\perp) = \frac{\hat{n}}{2\pi} \delta \left(H_\perp - \frac{\varepsilon^2}{2r_b^2} \right)$$

$$r_b = \left(\frac{Q + \sqrt{4k_{\beta 0}^2 \varepsilon^2 + Q^2}}{2k_{\beta 0}^2} \right)^{1/2} = \text{const}$$

$k_{\beta 0}$ =	Undepressed betatron wavenumber
r_b =	Beam edge radius
\hat{n} =	Beam number density
Q =	Dimensionless permeance
ε =	rms edge emittance

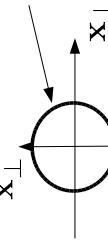
Further comments on the KV equilibrium: Distribution Structure

Equilibrium distribution:

$$f_\perp \sim \delta[\text{Courant-Snyder invariants}]$$

Forms a highly singular hyper-shell in 4D phase-space

Schematic:



- ♦ Singular distribution has large “Free-Energy” to drive many instabilities
 - Low order envelope modes are physical and highly important (sec: S.M. Lund, lectures on **Centroid and Envelope Descriptions of Beams**)
 - Perturbative analysis shows strong collective instabilities
 - Hofmann, Laslett, Smith, and Haber, Part. Accel. **13**, 145 (1983)
 - Higher order instabilities (collective modes) have unphysical aspects due to (delta-function) structure of distribution and must be applied with care (see following lecture material)
 - Instabilities can cause problems if the KV distribution is employed as an initial beam state in self-consistent simulations

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A full kinetic stability analysis of the KV equilibrium distribution is complicated and uncovers many strong instabilities

[I. Hofmann, J.L. Laslett, L. Smith, and I. Haber, Particle Accel. 13, 145 (1983);

R. Gluckstern, Proc. 1970 Proton Linac Conf., Batavia 811 (1971)]

Expand Vlasov's equation to linear order with: $f_{\perp}(C.S. \text{ Invariant}) = \text{equilibrium}$

$$f_{\perp} \rightarrow f_{\perp}(\text{C.S. Invariant}) + \delta f_{\perp}$$

Solve the Poisson equation:

$$\nabla_{\perp}^2 \delta\phi = -\frac{q}{\epsilon_0} \int d^2x' \delta f_{\perp}$$

using truncated polynomials for $\delta\phi$ internal to the beam to represent a "normal mode" with pure harmonic variation, i.e., $\delta\phi \sim \text{func}(x, y)e^{-iks}$

$$\delta\phi = \sum_{m=0}^n A_m^{(0)}(s) x^{n-m} y^m + \sum_{m=0}^{n-2} A_m^{(1)}(s) x^{n-m-2} y^m + \dots$$

$$n = 2, 3, 4, \dots$$

"order" of mode

m can be restricted to even or odd terms

◆ Truncated polynomials can meet all boundary conditions

◆ Eigenvalues of a Floquet form transfer matrix analyzed for stability properties

- Lowest order results reproduce KV envelope instabilities

- Higher order results manifest many strong instabilities

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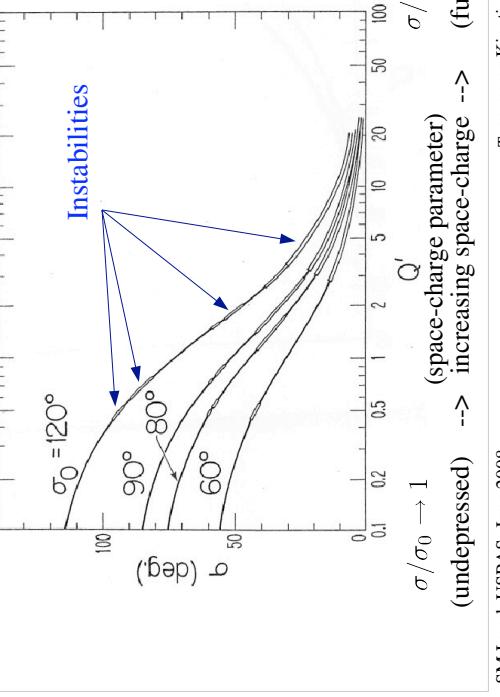
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Higher order kinetic instabilities of the KV equilibrium are strong and cover a wide parameter range for periodic focusing lattices

Example: FODO Quadrupole Stability

4th order ($n = 4$) even mode

[Hofmann et. al, Particle Accel. 13, 145 (1983)]



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The continuous focusing limit can be analyzed to better understand properties of internal modes on a KV beam (1)

[S. Lund and R. Davidson, Physics of Plasmas 5, 3028 (1998); see Appendix B, C]

Continuous focusing, symmetric beam:

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon$$

$$r_x = r_y \equiv r_b$$

Mode eigenfunction (2n "order" in the sense of Hoffman et. al.):

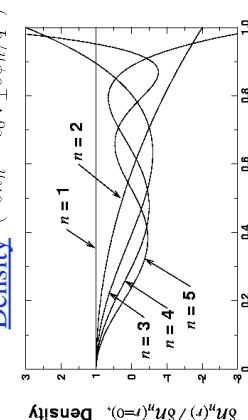
$$\delta\phi_n = \begin{cases} \frac{A_n}{2} \left[P_{n-1} \left(1 - 2\frac{r^2}{r_b^2} \right) + P_n \left(1 - 2\frac{r^2}{r_b^2} \right) \right], & 0 \leq r \leq r_b \\ 0, & r_b < r \end{cases}$$

$$A_n = \text{const}$$

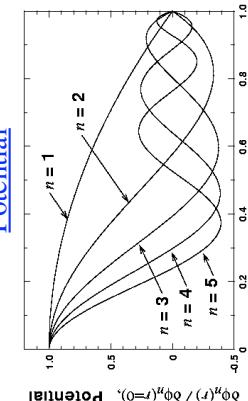
$$n = 1, 2, 3, \dots$$

$$P_n(x) = n^{\text{th}} \text{ order Legendre polynomial}$$

Density ($\delta n_n = \epsilon_0 \nabla_{\perp}^2 \delta\phi_n / q$)



Potential



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The continuous focusing limit can be analyzed to better understand properties of internal modes on a KV beam (2)

Mode dispersion relation for e^{-iks} variations:

$$2n + \frac{1 - \sigma/\sigma_0}{(\sigma/\sigma_0)^2} \left[B_{n-1} \left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0} \right) - B_n \left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0} \right) \right] = 0$$

$$\text{where: } B_j(\alpha) \equiv \begin{cases} 1, & \frac{(\alpha/2)^2 - 0^2}{(\alpha/2)^2 - 1^2} \frac{(\alpha/2)^2 - 2^2}{(\alpha/2)^2 - 2^2} \dots \frac{(\alpha/2)^2 - (j-1)^2}{(\alpha/2)^2 - j^2} \\ \frac{(\alpha/2)^2 - 1^2}{(\alpha/2)^2 - 1^2} \frac{(\alpha/2)^2 - 3^2}{(\alpha/2)^2 - 3^2} \dots \frac{(\alpha/2)^2 - (j-1)^2}{(\alpha/2)^2 - j^2} & j = 1, 3, 5, \dots \\ \frac{1}{(\alpha/2)^2 - 2^2} \frac{(\alpha/2)^2 - 4^2}{(\alpha/2)^2 - 4^2} \dots \frac{(\alpha/2)^2 - (j-1)^2}{(\alpha/2)^2 - j^2} & j = 2, 4, 6, \dots \end{cases}$$

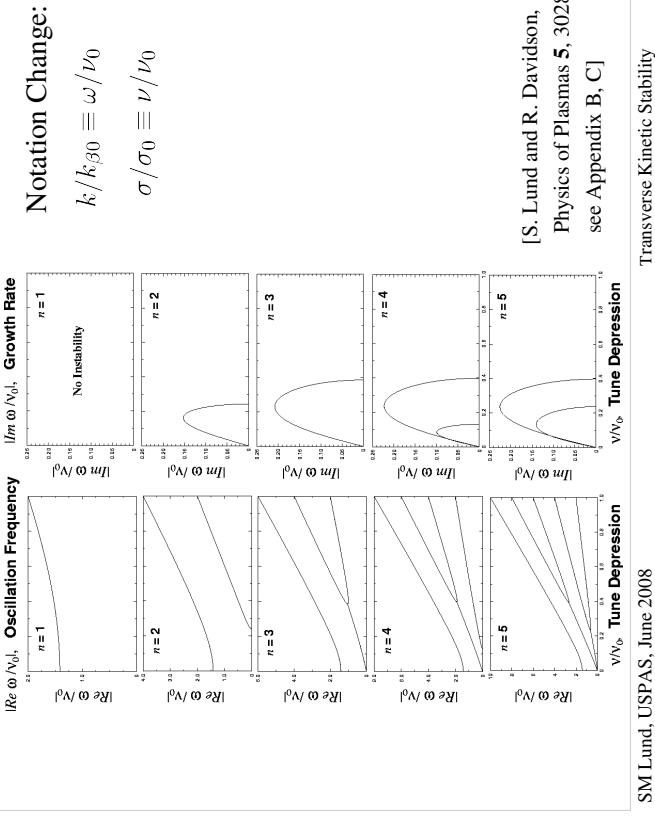
- ◆ Eigenfunction structure suggestive of wave perturbations often observed internal to the beam in simulations for a variety of beam distributions
- ◆ n distinct branches for 2n order (real coefficient) polynomial dispersion relation
- ◆ Some range of σ/σ_0 will be unstable for all $n > 1$
 - Instability exists for some n for $\sigma/\sigma_0 < 0.3985$
 - Growth rates are strong

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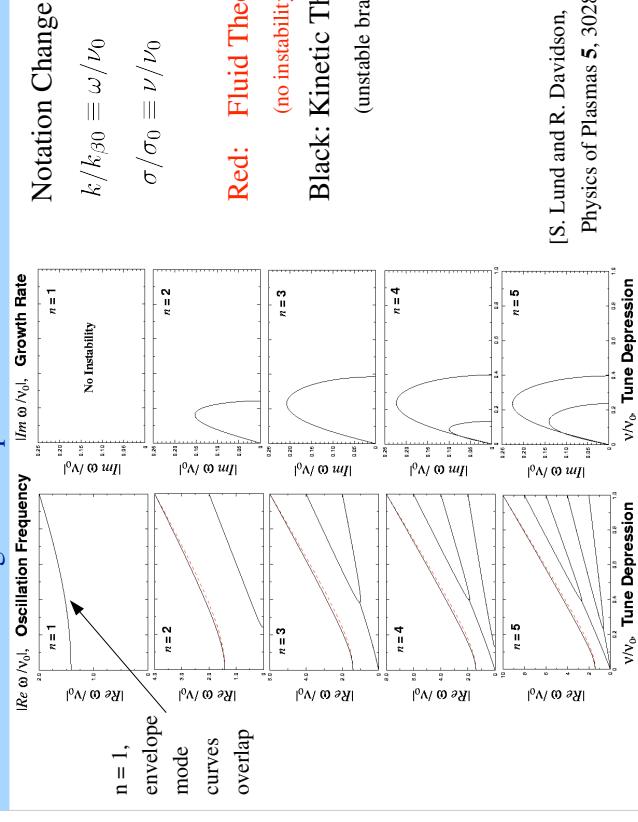
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Continuous focusing limit dispersion relation results for KV beam stability



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Continuous focusing limit dispersion relation results for KV beam stability



For continuous focusing, fluid theory shows that some branches of the KV dispersion relation *are* physical

[S. Lund and R. Davidson, Physics of Plasmas 5, 3028 (1998)]

Fluid theory:

- ◆ KV equilibrium distribution is reasonable in fluid theory
- No singularities
- Flat density and parabolic radial temperature profiles
- ◆ Theory truncated by assuming zero heat flow

Mode eigenfunctions:

Exactly the same as derived under kinetic theory!

Mode dispersion relation:

$$\frac{k}{k_{\beta 0}} = \sqrt{2 + 2 \left(\frac{\sigma}{\sigma_0} \right)^2 (2n^2 - 1)}$$

$$n = 1, 2, 3, \dots$$

Single, stable branch

- Agrees well with high frequency branch from kinetic theory

Results show that aspects of higher-order KV internal modes are physical!
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S5: Global Conservation Constraints

Apply for any initial distribution, equilibrium or not.

- ◆ Strongly constrain nonlinear evolution of the system.
- ◆ Valid even with a beam pipe provided that particles are not lost from the system and that symmetries are respected.
- ◆ Useful to bound perturbations, but yields no information on evolution timescales.

1) Generalized Entropy

$$U_G = \int d^2x_\perp \int d^2x'_\perp G(f_\perp) = \text{const}$$

$G(f_\perp) = \text{Any differentiable functions satisfying } G(f_\perp \rightarrow 0) = 0$

◆ Applies to all Vlasov evolutions.

// Examples
Line-charge: $G(f_\perp) = qf_\perp$ →

$$\text{Entropy: } G(f_\perp) = -\frac{f_\perp}{A} \ln \left(\frac{f_\perp}{f_0} \right) \quad A, f_0 \text{ constants}$$

$$\mathcal{S} = -\int \frac{d^2x}{A} \int d^2x' f_\perp \ln \left(\frac{f_\perp}{f_0} \right) = \text{const}$$

[S. Lund, USPAS, June 2008 Transverse Kinetic Stability 32]

2) Transverse Energy in continuous focusing

$$U_{\mathcal{E}} = \int d^2x'_\perp \int d^2x_\perp \left\{ \frac{1}{2}\mathbf{x}'_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 \right\} f_\perp + \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2} = \text{const}$$

Here,

$$\int d^2x'_\perp \int d^2x_\perp \frac{1}{2}\mathbf{x}'_\perp'^2 f_\perp \sim \text{Kinetic Energy}$$

$$\int d^2x'_\perp \int d^2x_\perp \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 f_\perp \sim \text{Potential Energy}$$

of applied focusing forces

$$\int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2} \sim \text{Self-Field Energy}$$

♦ Does not hold when focusing forces vary in s

- Can still be approximately valid for rms matched beams where energy will regularly pump into and out of the beam

♦ Self field energy term diverges in radially unbounded systems (no aperture)

♦ Still useful if an appropriate infinite constant is subtracted (to regularize)

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Comments on system energy form:

$$U_{\mathcal{E}} = \int d^2x'_\perp \int d^2x_\perp \left\{ \frac{1}{2}\mathbf{x}'_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 \right\} f_\perp + \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2m\gamma_b^3\beta_b^2c^2} = \text{const}$$

zero for grounded aperture

Analyze the energy term:

$$\int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int d^2x_\perp \frac{1}{2}\nabla_\perp \cdot (\phi \nabla_\perp \phi) - \int d^2x_\perp \frac{1}{2}\phi \nabla_\perp^2 \phi$$

or infinite constant

in free space

$$\nabla_\perp^2 \phi = -\frac{q}{\epsilon_0} \int d^2x'_\perp f_\perp$$

$$\text{Employ the Poisson equation: } \rightarrow \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int d^2x'_\perp \int d^2x_\perp \frac{q}{2\epsilon_0} \phi f_\perp$$

Giving:

$$U_{\mathcal{E}} = \int d^2x'_\perp \int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int d^2x_\perp \int d^2x'_\perp \frac{q}{2\epsilon_0} \phi f_\perp$$

- ♦ Note the relation to the system Hamiltonian with a symmetry factor to not double count particle contributions

$$H_\perp = \frac{1}{2}\mathbf{x}'_\perp^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{1}{2m\gamma_b^3\beta_b^2c^2} \frac{q\phi}{\epsilon_0} f_\perp = \text{const}$$

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Comments on self-field energy divergences:

In unbounded (free space) systems, far from the beam the field must look like a line charge:

$$-\frac{\partial\phi}{\partial r} \sim \frac{\lambda}{2\pi\epsilon_0 r} \quad r > r_{\text{large}}$$

Resolve the total field energy into a finite (near) term and a divergent term:

$$\int d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} = \int_{r \leq r_{\text{large}}} d^2x_\perp \frac{\epsilon_0|\nabla_\perp\phi|^2}{2} + \frac{\lambda^2}{4\pi\epsilon_0} \int_{r_{\text{large}}}^\infty dr \frac{1}{r}$$

total	finite term	logarithmically
		divergent term

- ♦ This divergence can be subtracted out to thereby regularized the system energy
- Renders energy constraint useful for application to equilibria in radially unbounded systems such as thermal equilibrium

- ♦ Trivial in present model, but useful when equations of motion are generalized to allow for a spread in axial momentum

3) Angular Momentum

$$U_\theta = \int d^2x_\perp \int d^2x'_\perp (yx' - x'y)f_\perp = \text{const}$$

- ♦ Focusing and beam pipe (if present) must be axisymmetric

- Useful for solenoidal magnetic focusing
- Does not apply to alternating gradient quadrupole focusing since such systems do not have the required axisymmetry

4) Axial Momentum

$$U_z = \int d^2x_\perp \int d^2x'_\perp m\gamma_b\beta_b c f_\perp = \text{const}$$

- ♦ Trivial in present model, but useful when equations of motion are generalized to allow for a spread in axial momentum

Comments on applications of the global conservation constraints:

Global invariants strongly constrain the nonlinear evolution of the system

- Only evolutions consistent with Vlasov's equation are physical
- Constraints consistent with the model can bound kinematically accessible evolutions

Application of the invariants does not require (difficult to derive) normal mode descriptions

- But cannot, by itself, provide information on evolution timescales

Use of global constraints to bound perturbations has appeal since distributions in real machines may be far from an equilibrium. Used to:

- Derive sufficient conditions for stability
- Bound particle losses [O'Neil, Phys. Fluids **23**, 2216 (1980)]
- Bound changes of system moments (for example the rms emittance) under assumed relaxation processes

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S6: Kinetic Stability Theorem for continuous focusing equilibria

[Fowler, J. Math Phys. **4**, 559 (1963); Gardner, Phys. Fluids **6**, 839 (1963); R. Davidson, Physics of Nonneutral Plasmas, Addison-Wesley (1990)]

Resolve:

$$\begin{aligned} f_{\perp} &= f_0(H_0) + \delta f_{\perp} \\ f_0(H_0) &= \text{Equilibrium (subscript 0) distribution} \\ \delta f_{\perp} &= \text{Perturbation about equilibrium} \end{aligned}$$

Employ generalized entropy and transverse energy global constraints (S5):

$$U_G = \int d^2x_{\perp} \int d^2x'_{\perp} G(f_{\perp}) = \text{const}$$

$$U_{\mathcal{E}} = \int d^2x'_{\perp} \int d^2x_{\perp} \left\{ \frac{1}{2}\mathbf{x}_{\perp}'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_{\perp}^2 \right\} f_{\perp} + \int d^2x_{\perp} \frac{\epsilon_0 |\nabla_{\perp} \phi|^2}{2m\gamma_b^3 \beta_b^2 c^2} = \text{const}$$

Apply to equilibrium and full distribution to form an effective "free-energy":

$$\Delta U_G = U_G - U_{G0} = \text{const}$$

$$\begin{aligned} F &= \Delta U_{\mathcal{E}} - \Delta U_G \\ &= \int d^2x'_{\perp} \int d^2x_{\perp} \left\{ \frac{1}{2}\mathbf{x}_{\perp}'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_{\perp}^2 \right\} \delta f_{\perp} + \int d^2x_{\perp} \frac{\epsilon_0 |\nabla_{\perp} \phi|^2}{2m\gamma_b^3 \beta_b^2 c^2} \\ &\quad + \int d^2x_{\perp} \int d^2x'_{\perp} [G(f_{\perp}) - G(f_0)] = \text{const} \end{aligned}$$

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The perturbed potential satisfies:

$$\delta\phi \equiv \phi - \phi_0 \quad \nabla_{\perp}^2 \delta\phi = -\frac{q}{\epsilon_0} \int d^2x'_{\perp} \delta f_{\perp}$$

Take $|\delta f_{\perp}| \ll f_0$ and Taylor expand to 2nd order

$$G(f_0 + \delta f_{\perp}) = G(f_0) + \frac{dG(f_0)}{df_0} \delta f_{\perp} + \frac{d^2G(f_0)}{df_0^2} \frac{(\delta f_{\perp})^2}{2} + \Theta(\delta^3)$$

Without loss of generality, choose:

$$\frac{dG(f_0)}{df_0} = -H_0 = -\left(\frac{1}{2}\mathbf{x}_{\perp}'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_{\perp}^2 + \frac{q\phi}{m\gamma_b^2 \beta_b^2 c^2}\right)$$

◆ This choice can always be realized

Then $\frac{d^2G(f_0)}{df_0^2} = -\frac{\partial H_0}{\partial f_0} = -\frac{-1}{\partial f_0(H_0)/\partial H_0}$

Some steps (few lines using partial integration) yields:

$$F = \int d^2x_{\perp} \left\{ \frac{\epsilon_0 |\nabla_{\perp} \delta\phi|^2}{2m\gamma_b^3 \beta_b^2 c^2} - \int d^2x'_{\perp} \frac{(\delta f_{\perp})^2}{\partial f_0(H_0)/\partial H_0} \right\} + \Theta(\delta^3) = \text{const}$$

- ◆ If $\partial f_0(H_0)/\partial H_0 < 0$ then F is a sum of two positive definite terms and perturbations are bounded by $F = \text{const.}$

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Kinetic Stability Theorem

If $f_{\perp}(H_{\perp})$ is a monotonic decreasing function of H_{\perp} with $\partial f_{\perp}(H_{\perp})/\partial H_{\perp} < 0$ then the equilibrium defined by $f_{\perp}(H_{\perp})$ is stable to arbitrary small-amplitude perturbations.

- ◆ Is a sufficient condition for stability
- Equilibria that violate the theorem may or may not be stable
- ◆ Mean value theorem can be used to generalize conclusions for arbitrary amplitude
- R. Davidson proof

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// Example Applications of Kinetic Stability Theorem

KV Equilibrium:

$$f_{\perp}(H_{\perp}) = \frac{\hat{n}}{2\pi} \delta(H_{\perp} - H_{\perp b})$$

$\partial f_{\perp}/\partial H_{\perp}$ changes sign

- ♦ Full normal mode analysis in Kinetic theory shows strong instabilities when space-charge becomes strong
- ♦ Not surprising, delta function represents a highly inverted population in phase-space with “free-energy” to drive instabilities

Waerberg Equilibrium:

$$f_{\perp}(H_{\perp}) = f_0 \Theta(H_{\perp b} - H_{\perp})$$

$$\frac{\partial f_{\perp}}{\partial H_{\perp}} = f_0 \delta(H_{\perp} - H_{\perp b}) \leq 0$$

monotonic decreasing, stable by theorem

Thermal Equilibrium:

$$f_{\perp}(H_{\perp}) = f_0 \exp(-\beta H_{\perp}),$$

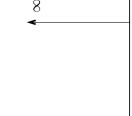
$$\frac{\partial f_{\perp}}{\partial H_{\perp}} = -f_0 \beta \exp(-\beta H_{\perp}) \leq 0$$

monotonic decreasing, stable by theorem

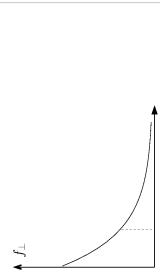
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“free-energy” to drive instabilities



monotonic decreasing, stable by theorem

- Add material to discuss combined application of the density inversion theorem and the kinetic stability theorem
 - ♦ Monotonic decreasing radial density profile $n(r)$ gives monotonic decreasing distribution $f(H)$
 - ♦ Stability of radial density profiles follows for continuous focusing
 - ♦ Extent this can be generalized to periodic focusing?

S7: rms Emittance Growth and Nonlinear Forces

Fundamental theme of beam physics is to minimize statistical beam emittance growth in transport to preserve focusability on target

Return to the full transverse beam model with:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x} + \text{Applied Nonlinear Field Terms}$$

and express as:

$$x''(s) + \kappa_x(s)x(s) = f_x^L(s)x(s) + F_x^{NL}(x, y, s)$$

$f_x^L(s) = \text{Linear Space-Charge Coefficient}$

$$F_x^{NL}(x, y, s) = \begin{cases} \text{Nonlinear Forces + Linear Skew Coupled Forces} \\ \text{(Applied and Space-Charge)} \end{cases}$$

// Examples:

$$f_x^L(s) = \frac{Q}{r_b(s)} \quad \begin{array}{l} \text{Self-field forces within an axisymmetric (mismatched) KV} \\ \text{beam core in a continuous focusing model} \end{array}$$

$$F_x^{NL}(x, y, s) = \text{Im} \left[\frac{b_3}{r_p} \left(\frac{x+iy}{r_p} \right)^2 \right] \quad \begin{array}{l} \text{Electric (with normal and skew components)} \\ \text{sextupole optic based on multipole expansions} \\ \text{(see: lectures on Particle Equations of Motion)} \end{array}$$

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From the definition of the statistical (rms) emittance:

$$\varepsilon_x \equiv 4[\langle x^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle xx' \rangle_{\perp}^2]^{1/2}$$

Differentiate the squared emittance and apply the chain rule:

$$\begin{aligned} \frac{d}{ds} \varepsilon_x^2 &\equiv 32[\langle xx' \rangle_{\perp} \langle x'^2 \rangle_{\perp} + \langle x^2 \rangle_{\perp} \langle x'x'' \rangle_{\perp} - \langle xx' \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle xx' \rangle_{\perp} \langle xx'' \rangle_{\perp}] \\ &= 32[\langle x^2 \rangle_{\perp} \langle x'x'' \rangle_{\perp} - \langle xx' \rangle_{\perp} \langle xx'' \rangle_{\perp}] \end{aligned}$$

Insert the equations of motion:

$$x'' + \kappa_x x = f_x^L x + F_x^{NL}$$

The linear terms cancel to show *for any beam distribution* that:

$$\frac{d}{ds} \varepsilon_x^2 = 32 [\langle x^2 \rangle_{\perp} \langle x'x'' \rangle_{\perp} - \langle xx' \rangle_{\perp} \langle x'F_x^{NL} \rangle_{\perp}]$$

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Implications of:

$$\frac{d}{ds}\varepsilon_x^2 = 32 [\langle x^2 \rangle_\perp \langle x' F_x^{NL} \rangle_\perp - \langle xx' \rangle_\perp \langle x F_x^{NL} \rangle_\perp]$$

◆ Emittance evolution/growth is driven by nonlinear or skew coupling forces

- Nonlinear terms can result from applied or space-charge fields

- More detailed analysis shows that skew coupled forces

cause x-y plane transfer oscillations but there is still a 4D quadratic invariant

◆ Minimize nonlinear forces to preserve emittance and maintain focusability

◆ This result (essentially) has already been demonstrated in the problem sets for the

Introductory Lectures

If the beam is accelerating, the equations of motion become:

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + \kappa_x x = f_x^L x + F_x^{NL}$$

and this result can be generalized (see homework problems) using the normalized emittance:

$$\varepsilon_{nx} \equiv \gamma_b \beta_b \varepsilon_x$$

$$\frac{d}{ds}\varepsilon_{nx}^2 = 32(\gamma_b \beta_b)^2 [\langle x^2 \rangle_\perp \langle x' F_x^{NL} \rangle_\perp - \langle xx' \rangle_\perp \langle x F_x^{NL} \rangle_\perp]$$

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S8: rms Emittance Growth and Nonlinear Space-Charge Forces

[Wangler et. al, IEEE Trans. Nucl. Sci. 32, 2196 (1985), Reiser, *Charged Particle Beams*, (1994)]

In continuous focusing all nonlinear force terms are from space-charge, giving:

$$\frac{d}{ds}\varepsilon_x^2 = -\frac{32q}{m\gamma_b^3\beta_b^2c^2} \left[\langle x^2 \rangle_\perp \langle x' \frac{\partial\phi}{\partial x} \rangle_\perp - \langle xx' \rangle_\perp \langle x \frac{\partial\phi}{\partial x} \rangle_\perp \right]$$

For any axisymmetric beam it can be shown that:

$$\langle x \frac{\partial\phi}{\partial x} \rangle_\perp = \frac{1}{2} \langle \frac{\partial\phi}{\partial r} \rangle_\perp = -\frac{\lambda}{8\pi\epsilon_0} \\ \langle x' \frac{\partial\phi}{\partial x} \rangle_\perp = \frac{1}{2} \langle r' \frac{\partial\phi}{\partial x} \rangle_\perp = \frac{1}{8\pi\epsilon_0\lambda} \frac{dW}{ds}$$

For any axisymmetric beam it can also be shown that:

$$\langle xx' \rangle_\perp = \frac{1}{2} \langle rr' \rangle_\perp = -\frac{\langle x^2 \rangle_\perp}{\lambda^2} \frac{dW_u}{ds} \\ W_u = \mathbf{W} \text{ for an rms equivalent uniform density beam}$$

These results give (Wangler, Lapostolle):

$$\frac{d}{ds}\varepsilon_x^2 = -4Q \langle x^2 \rangle_\perp \frac{d}{ds} \left(\frac{W - W_u}{\lambda^2} \right)$$

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Consider the rms envelope equation to better understand what is required for $r_b^2 = \text{const}$

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

◆ Valid in an rms equivalent sense with $\varepsilon \neq \text{const}$ for a non-KV beam

If the emittance term is small relative to the permeance term and the initial beam starts out as matched we can approximate the equation as

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

and it is reasonable to expect the beam radius to remain nearly constant under modest changes in emittance. This ordering must be checked after estimating the emittance change based the final to initial state energy differences. See S9 and S10 analysis for a better understanding on how this can be valid.

If the rms beam radius does not change much in the beam evolution:

$$r_b^2 = 2\langle x^2 \rangle_\perp \simeq \text{const}$$

Then the equation can be trivially integrated to show that:

$$\Delta_{fi}(\varepsilon_x^2) = -4Q r_b^2 \Delta_{fi} \left(\frac{W - W_u}{\lambda^2} \right) \\ \Delta_{fi}(\dots) \equiv \text{Final State Value} - \text{Initial State Value}$$

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S9: Uniform Density Beams and Extreme Energy States

Construct minima of the self-field energy per unit axial length:

$$W = \frac{\epsilon_0}{2} \int d^2x_\perp |\nabla_\perp \phi|^2$$

subject to:

$$\lambda = \text{const} \quad \dots \text{fixed line-charge}$$

$$r_b = \sqrt{2\langle r^2 \rangle_\perp} = \text{const} \quad \dots \text{fixed rms equivalent beam radius}$$

Using the method of Lagrange multipliers, vary (Helmholtz free energy):

$$F = W - \mu(\lambda/q)\langle r^2 \rangle_\perp = \int d^2x_\perp \left\{ \epsilon_0 \frac{|\nabla_\perp \phi|^2}{2} - \mu r^2 n \right\} \quad \mu = \text{const}$$

and require that variations satisfy the Poisson equation and conserve charge

$$\nabla_\perp^2 \delta\phi = -\frac{q}{\epsilon_0} \delta n \quad \delta\phi|_{\text{boundary}} = 0$$

Then variations terminate at 2nd order giving:

$$\delta F = - \int d^2x_\perp \{ \mu r^2 + \text{const} \} \delta n + \epsilon_0 \int d^2x_\perp \nabla_\perp \phi \cdot \nabla_\perp \delta\phi + \frac{\epsilon_0}{2} \int d^2x_\perp |\nabla_\perp \delta\phi|^2$$

Integrating the 2nd term by parts and employing the Poisson equation then gives:

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$$\delta F = \int d^2x_\perp \{ q\phi - \mu r^2 - \text{const} \} \delta n + \frac{\epsilon_0}{2} \int d^2x_\perp |\nabla_\perp \delta\phi|^2$$

For an extremum, the first order term must vanish, giving *within the beam*:

$$q\phi = \mu r^2 + \text{const}$$

From Poisson's equation:

$$\frac{1}{r} \left(r \frac{\partial \phi}{\partial r} \right) \phi = \text{const}$$

This is the density of a uniform, axisymmetric beam, which implies that a uniform density axisymmetric beam results in an extremum. This extremum is also a global maximum since all variations about it (2nd term of boxed equation above) are positive definite.

Result:

At fixed line charge and rms radius, a uniform density beam minimizes the electrostatic self-field energy

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S10: Collective Relaxation and rms Emittance Growth

The space-charge profile of intense beams can be born highly nonuniform out of nonideal (real) injectors or become nonuniform due to a variety of (error) processes. Also, low-order envelope matching of the beam may be incorrect due to focusing and/or distribution errors.

How much emittance growth and changes in other characteristic parameters may be induced by relaxation of characteristic perturbations?

- ♦ Employ Global Conservation Constraints of system to bound possible changes
- ♦ Assume full relaxation to a final, uniform density state for simplicity
- ♦ What is the mechanism for the assumed relaxation?
- ♦ Collective modes launched by errors will have a broad spectrum
 - Phase mixing can smooth nonuniformities – mode frequencies incommensurate
 - Nonlinear interactions, Landau damping, interaction with external errors, ...
 - Certain errors more/less likely to relax:
- Internal wave perturbations expected to relax due to many interactions
 - Envelope mismatch will not (coherent mode) unless amplitudes are very large producing copious halo and nonlinear interactions

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The result:

At fixed line charge and rms radius, a uniform density beam minimizes the electrostatic self-field energy

combined with Wangler's Theorem:

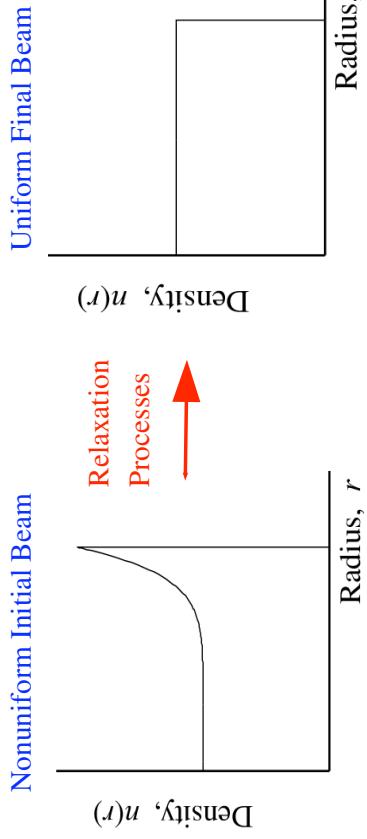
$$\frac{d}{ds} \varepsilon_x^2 = -Q \langle x^2 \rangle_\perp \frac{d}{ds} \left(\frac{W - W_u}{\lambda^2} \right)$$

shows that:

- ♦ Self-field energy changes from beam nonuniformity drives emittance evolution
- ♦ Expect the following trends in an evolving beam density profile
 - *Nonuniform* density => *more uniform* density <=> local emittance *growth*
 - *Uniform* density => *more nonuniform* density <=> local emittance *reduction*
 - ♦ Should attempt to maintain beam density uniformity to preserve beam emittance and focusability
- Results can be partially generalized to 2D elliptical beams
 - [J. Struckmeier and I. Hofmann, Part Accel. **39**, 219 (1992)]

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Example: Relaxation of nonlinear space-charge waves



Reference: High resolution self-consistent PIC simulations shown in class

- ♦ Continuous focusing and a more realistic FODO transport lattice
 - Relaxation more complete in real lattice due to a richer frequency spectrum
- ♦ Relaxations surprisingly rapid: few undepressed betatron wavelengths observed in simulations

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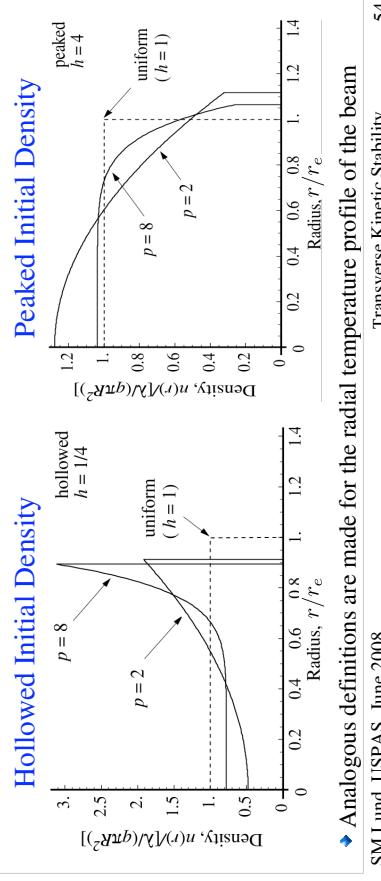
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Initial Nonuniform Beam Parameterization

$$n(r) = \begin{cases} \hat{n} \left[1 + \frac{1-h}{h} \left(\frac{r}{r_e} \right)^p \right], & 0 \leq r \leq r_e \\ 0, & r_e < r \leq r_p \end{cases}$$

$$\lambda = \int d^2x_\perp n = \pi q\hat{n}r_e^2 \left[\frac{(ph+2)}{(p+2)h} \right]$$

Normalize profiles to compare common rms radius (r_b) and total charge (λ)

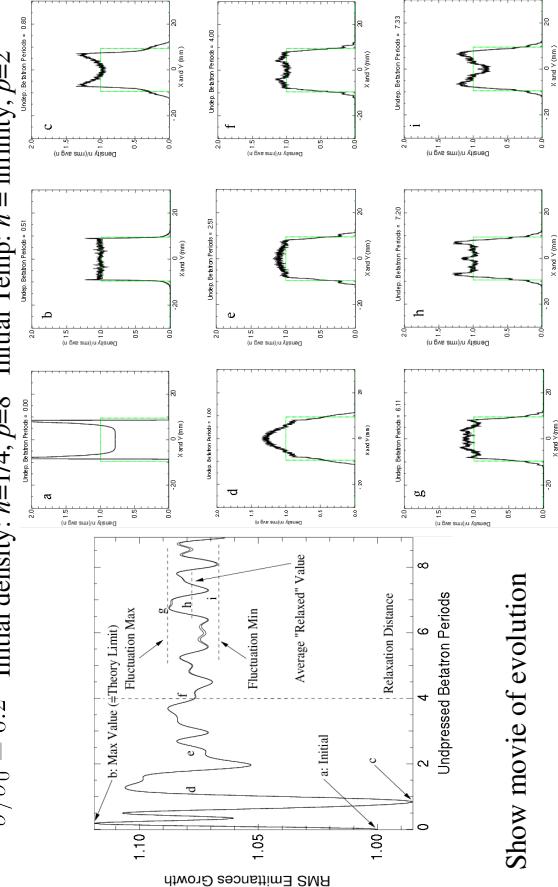


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Example Simulation, Initial Nonuniform Beam

$\sigma/\sigma_0 = 0.2$ Initial density: $h=1/4, p=8$ Initial Temp: $h = \infty$, $p=2$



Show movie of evolution

[Lund, Grote, and Davidson, Nucl. Instr. Meth. A 544, 472 (2005)]

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Simulation results for a broad range of strong space-charge

Initial beam	Density		Temperature		Theory	Simulation
	h	p	h	p		
0.1	0.25	4	1	arb.	1.57	1.42 (1.57, 1.31-1.52)
			∞	2		1.45 (1.57, 1.38-1.52)
0.25	0.25	8	0.5	1	1.43	1.41 (1.57, 1.30-1.52)
			∞	2		1.33 (1.43, 1.28-1.38)
0.5	0.5	8	0.5	1	1.43	1.35 (1.43, 1.30-1.40)
			∞	2		1.32 (1.43, 1.26-1.38)
0.20	0.25	4	1	arb.	1.17	1.11 (1.16, 1.09-1.13)
			∞	2		1.12 (1.16, 1.10-1.13)
0.25	0.25	8	0.5	1	1.12	1.08 (1.16, 1.09-1.13)
			∞	2		1.08 (1.12, 1.06-1.09)
0.5	0.5	8	0.5	1	1.08	1.08 (1.12, 1.06-1.09)
			∞	2		1.08 (1.12, 1.06-1.09)

Theory results based on conservation of system charge and energy used to calculate the change in rms edge radius between initial (i) and final (f) matched beam states

$$\frac{(r_{bf}/r_{bi})^2 - 1}{1 - (\sigma_i/\sigma_0)^2} + \frac{p(1 - h)[4 + p + (3 + p)h]}{(p + 2)(p + 4)(2 + ph)^2} - \ln \left[\sqrt{\frac{(p + 2)(ph + 4)}{(p + 4)(ph + 2)}} \frac{rbf}{rb_i} \right] = 0$$

Ratios of final to initial emittance are then obtainable from the matched envelope eqns:

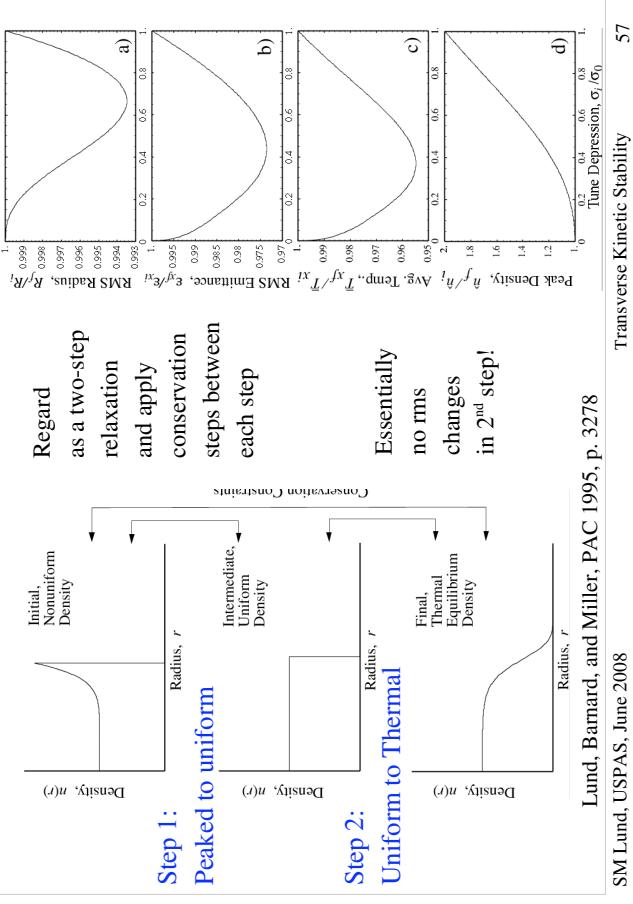
$$\frac{\varepsilon_x f}{\varepsilon_x i} = \frac{rbf}{rbi} \sqrt{\frac{(r_{bf}/r_{bi})^2 - [1 - (\sigma_i/\sigma_0)^2]}{(\sigma_i/\sigma_0)^2}}$$

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Theory estimates from global conservation constraints work well. What changes if the beam relaxes to a smooth thermal equilibrium instead? – Very little change



S11: Halo Induced Mechanism of Higher Order Instability

In periodic focusing with alternating gradient quadrupole focusing (most common case), it has been observed in simulations and the laboratory that good transport in terms of **little lost particles** or emittance growth is obtained when the applied focusing strength satisfies:

$$\sigma_0 \lesssim 85^\circ$$

little dependence on σ/σ_0

It has been a 40+ year unsolved problem by what primary mechanism this limit comes about. It was long thought that collective modes coupled to the lattice were responsible. However:

- ♦ Modes carry little free energy (see S10) to drive strong emittance growth
 - ♦ Particle losses and strong halo observed when stability criterion is violated
 - ♦ Collective internal modes likely also pumped but hard to explain with KV
- Recent progress helps clarify how this limit comes about via a strong halo-like resonance mechanism affecting near edge particles
- ♦ Does *not* require an equilibrium core beam

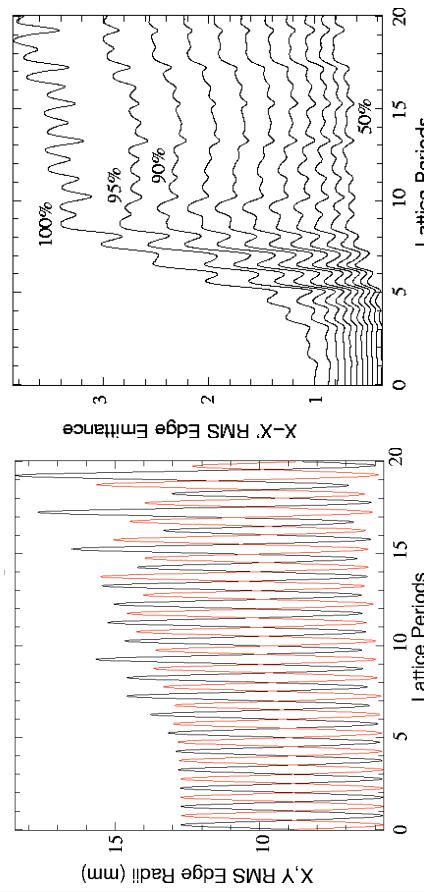
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Parametric PIC simulations of quadrupole transport agree with experimental observations and show that large rms emittance growth can occur rapidly

Parameters: $\sigma_0 = 110^\circ$, $\sigma/\sigma_0 = 0.2$ ($L_p = 0.5$ m, $\eta = 0.5$)

for initial semi-Gaussian distribution

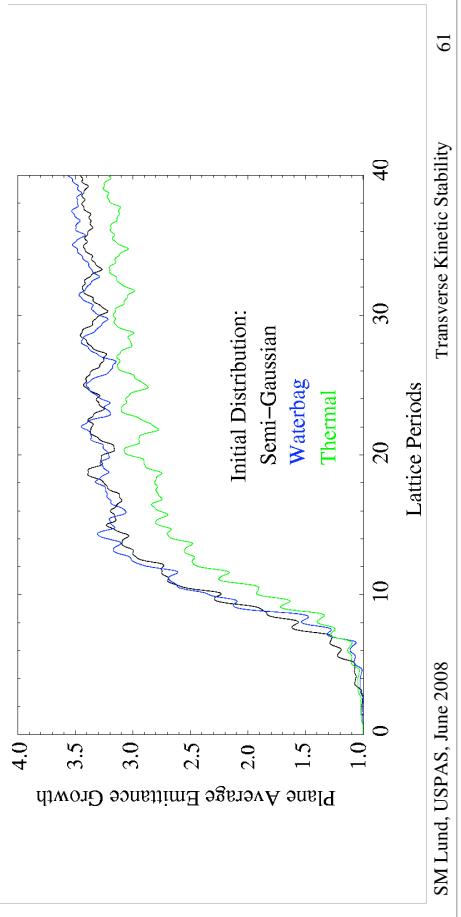


Higher $\sigma_0 \lesssim 85^\circ$ makes the onset of emittance growth larger and more rapid

Simulations suggest that transport limits observed are relatively insensitive to the structure of the initial distribution

Parameters: $\sigma_0 = 110^\circ$, $\sigma/\sigma_0 = 0.2$ ($L_p = 0.5$ m, $\eta = 0.5$)

- Wide class of initial distributions probed – little difference in x-y plane averages which help average over initial phase choices associated with launching conditions
- Growth becomes larger and faster with increasing σ_0

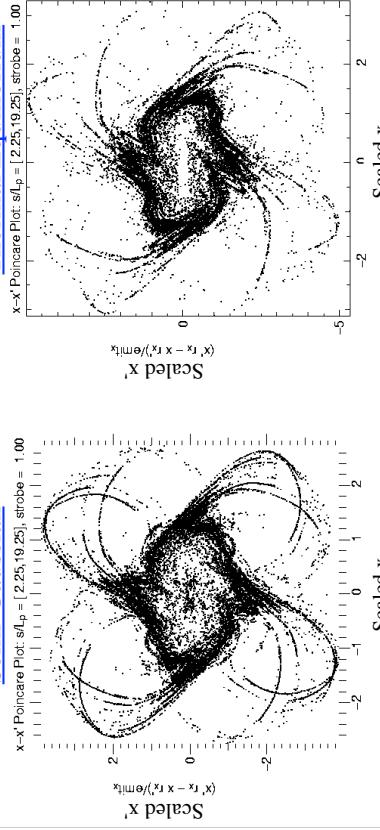


Poincaré plots generated from different initial distributions agree qualitatively in areas of strong instability and show large oscillation amplitude particle are halo like with resonant structure

Lattice period Poincaré strobe

$$\sigma_0 = 110^\circ \quad \sigma/\sigma_0 = 0.2$$

Semi-Gaussian Thermal Equilibrium

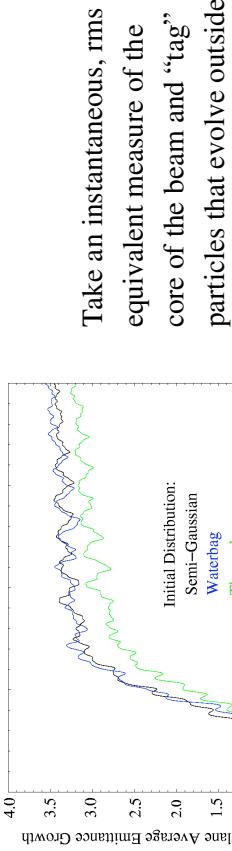


- Particles evolving along x-axis particles accumulated to generate clearer picture

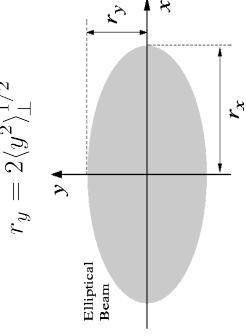
- Including off axis particles does *not* change basic conclusions

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An essential feature is that particles evolve outside the core of the beam



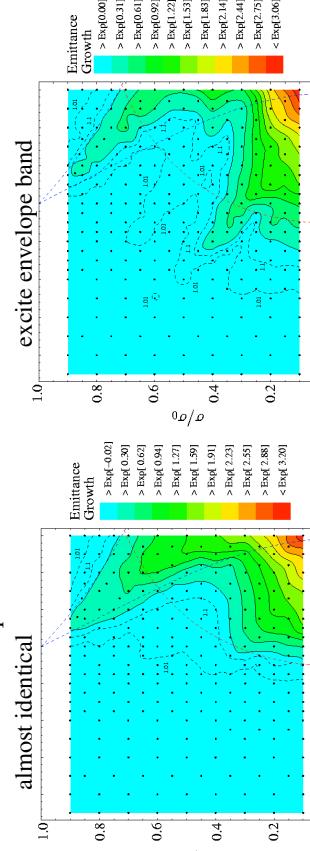
Take an instantaneous, rms equivalent measure of the core of the beam and “tag” particles that evolve outside the core:



Extensive simulations were carried out to better understand the parametric nature of the emittance growth

- All simulations carried out 6 undepressed betatron periods
 - Enough to resolve transition boundary: transition growth can be larger if run longer
 - Strong growth regions of initial distributions all similar (threshold can vary)
 - Irregular grid contouring with ~200 points (dots) to thoroughly probe possible instabilities
- Initial KV similar but does not excite envelope band

initial Waterbag Pseudo-Equilibrium



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Transverse Kinetic Stability 64

Core-Particle Model — Transverse particle equations of motion for a test particle moving inside and outside a uniform density elliptical beam envelope

$$x'' + \kappa_x x = \frac{2QF_x}{(r_x + r_y)r_x} x$$

$$y'' + \kappa_y y = \frac{2QF_y}{(r_x + r_y)r_y} y$$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^2 \beta_b^2 c^2} \quad \dots \quad \text{dimensionless permeance}$$

Where:

$$F_x = 1$$

$$F_y = 1$$

with

$$\tilde{S} \equiv \frac{\tilde{z}}{r_x^2 - r_y^2} [1 - \sqrt{1 - \frac{(r_x^2 - r_y^2)}{\tilde{z}^2}}]$$

$$F_x = (r_x + r_y) \frac{r_x'}{x} \text{Re}[\tilde{S}]$$

$$F_y = -(r_x + r_y) \frac{r_y'}{y} \text{Im}[\tilde{S}]$$

$$\tilde{z} = x + iy$$

$$i = \sqrt{-1}$$

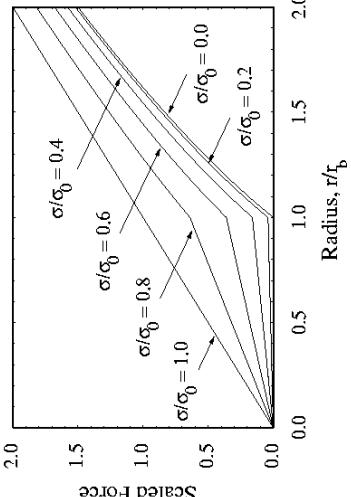
$$= \frac{1}{2\tilde{z}} \left[1 + \frac{1}{2} \frac{r_x^2 - r_y^2}{\tilde{z}^2} + \frac{1}{8} \frac{(r_x^2 - r_y^2)^2}{\tilde{z}^4} + \dots \right]$$

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Particles oscillating radially outside the beam envelope will experience oscillating nonlinear forces that vary with space-charge intensity and can drive resonances

Continuous Focusing Axisymmetric Beam Radial Force



- Nonlinear force transition at beam edge larger for strong space-charge
- Edge oscillations of matched beam enhance nonlinear effects acting on particles moving outside the envelope
- In AG focusing envelope oscillation amplitude scales strongly with

Transverse Kinetic Stability 66

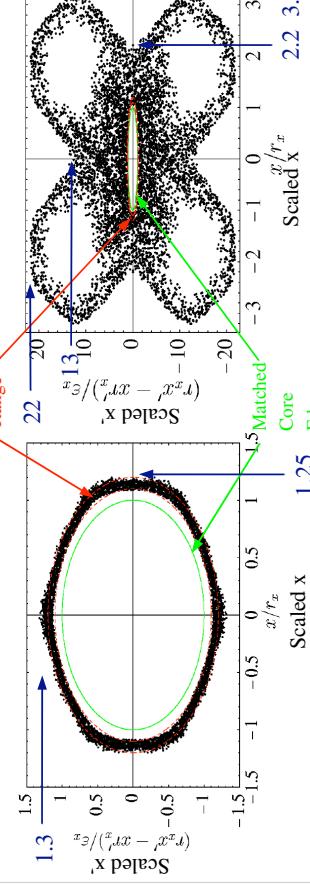
Core-particle simulations: Poincare plots illustrate resonances associated with higher-order halo production near the beam edge for FODO quadrupole transport

- High order resonances near the core are strongly expressed
- Resonances stronger for higher σ_0 and stronger space-charge
- Can overlap and break-up (strong chaotic transition) allowing particles launched *near the core* to rapidly increase in oscillation amplitude

Lattice Period Poincare Strobe, particles launched [1.1,1.2] times core radius

Stable **Unstable**

$\sigma_0 = 95^\circ, \sigma/\sigma_0 = 0.67$ $\sigma_0 = 110^\circ, \sigma/\sigma_0 = 0.1$



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Core-particle simulations: Amplitude pumping of characteristic “unstable” phase-space structures is typically rapid and saturates whereas stable cases

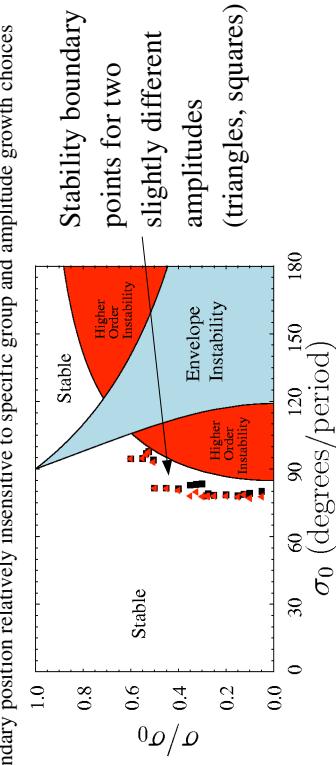
- $\sigma_0 = 60^\circ, \sigma/\sigma_0 = 0.1$
- $\sigma_0 = 110^\circ, \sigma/\sigma_0 = 0.1$
- $\sigma_0 = 110^\circ, \sigma/\sigma_0 = 0.1$
- Matched Beam
- Beam Initial Load Range
- Matched Beam
- Beam Initial Load Range
- Matched Beam
- Beam Initial Load Range

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Core particle simulations: Stability boundary data from a “halo” stability criterion agree with experimental data for quadrupole transport limits

- ♦ Start at a point (σ_0, σ) deep within the stable region
- ♦ While increasing σ_0 vary σ to find a point (if it exists) where initial launch groups [1.05, 1.10] outside the matched beam envelope are pumped to max amplitudes of 1.5 times the matched envelope
- Boundary position relatively insensitive to specific group and amplitude growth choices



Other halo analyses of transport limits conclude overly restrictive limits:

[Lagniel, Nuc. Instr. Meth. A **345**, 405 (1994)]

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Conclusions

High-order space-charge related emittance growth has long been observed in intense beam transport in quadrupole focusing channels with $\sigma_0 \gtrsim 85^\circ$:

- ♦ SBT Experiment at LBNL [M.G. Tiefenback, Ph.D Thesis, UC Berkeley (1986)]
- ♦ Simulations

A core-particle model has been developed that suggests observed transport limits result from a halo like mechanism:

- ♦ Near edge particles feel strong, rapidly oscillating nonlinear forces when moving just outside the matched beam envelope
- ♦ Drives a strongly chaotic resonance chain that limits at large amplitude resulting in a distorted beam and large statistical rms emittance growth
- ♦ Lack of core equilibrium provides a natural pump of significant numbers of particles outside the statistical beam edge and increase in oscillation amplitude

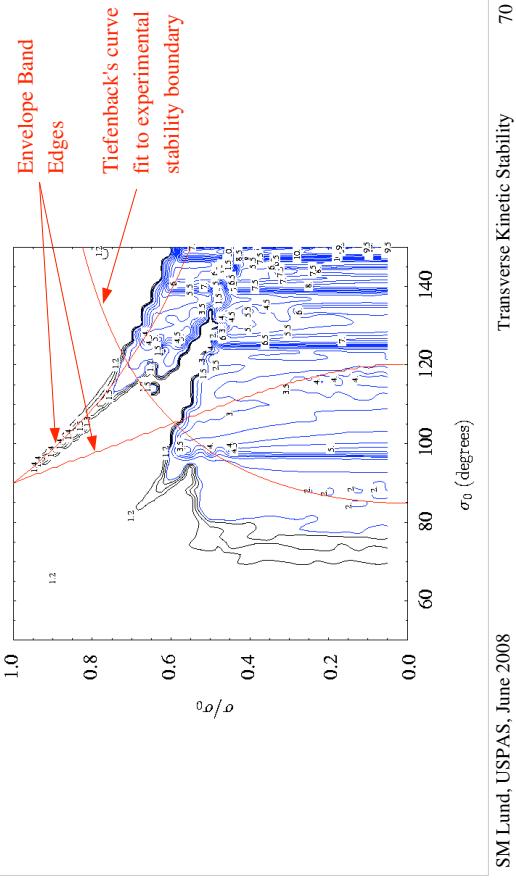
Instability mechanism expected to explain other features

- ♦ Stronger with envelope mismatch: consistent with observation that mismatched beams more unstable
- ♦ Weaker for focusing without much envelope fluctuation: high occupancy solenoids

Contours of maximum particle amplitudes obtained in the core particle model are strongly suggestive of trends observed in self-consistent simulation and experiment data

- ♦ Max amplitudes achieved for particles launched [11.05, 11.1] times the core radius:

- Variation with small changes in launch position small



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More Details:

Lund and Chawla, *Space-charge transport limits of ion beams in periodic quadrupole focusing channels, Nuc. Instr. Meth. A* **561**, 203 (2006)

Lund, Barnard, Bulkh, Chawla, and Chilton, *A core-particle model for periodically focused ion beams with intense space-charge, Nuc. Instr. Meth. A* **577**, 173 (2006)

Lund, Kikuchi, and Davidson, *Generation of initial kinetic distributions for simulation of long-pulse charged particle beams with high space-charge intensity*, submitted to PRSTAB

S12: Phase Mixing and Landau Damping in Beams

To be covered in future editions of class notes

- ♦ Likely inadequate time in lectures

These slides will be corrected and expanded for reference and any future editions of the US Particle Accelerator School class:
Beam Physics with Intense Space Charge, by J.J. Barnard and S.M. Lund

Corrections and suggestions are welcome. Contact:

Steven M. Lund
Lawrence Berkeley National Laboratory
BLDG 47 R 0112
1 Cyclotron Road
Berkeley, CA 94720-8201

SMLund@lbl.gov
(510) 486 – 6936

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References: For more information see:

- M. Reiser, *Theory and Design of Charged Particle Beams*, Wiley (1994)
R. Davidson, *Theory of Nonneutral Plasmas*, Addison-Wesley (1989)
R. Davidson and H. Qin, Physics of Intense Charged Particle Beams in High Energy Accelerators, World Scientific (2001)
F. Sacherer, *Transverse Space-Charge Effects in Circular Accelerators*, Univ. of California Berkeley, Ph.D Thesis (1968)
S. Lund and B. Bulkh, Review Article: *Stability Properties of the Transverse Envelope Equations Describing Intense Beam Transport*, PRST-Accel. and Beams 7, 024801 (2004)
S. Lund and R. Davidson, *Warm Fluid Description of Intense Beam Equilibrium and Electrostatic Stability Properties*, Phys. Plasmas 5, 3028 (1998)
D. Nicholson, *Introduction to Plasma Theory*, Wiley (1983)

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- Lund and Chawla, *Space-charge transport limits of ion beams in periodic quadrupole focusing channels*, Nuc. Instr. Meth. A **561**, 203 (2006)
Lund, Barnard, Bulkh, Chawla, and Chilton, *A core-particle model for periodically focused ion beams with intense space-charge*, Nuc. Instr. Meth. A **577**, 173 (2006)

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§4 Collective Modes on a KV's Equilibrium Beam

Here we take a KV equilibrium distribution with

$$f_0(H_0) = \frac{\hat{n}}{2\pi} \delta\left[H_0 - \frac{E_x^2}{2\Gamma_b^2}\right]$$

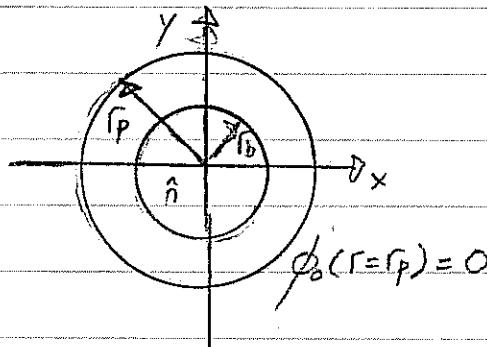
\hat{n} = constant density of KV equilibrium

E_x^2 = x -emittance.

Γ_b = equilibrium beam radius.

$$\frac{E_{B0}^2}{\Gamma_b} \Gamma_b - \frac{Q}{\Gamma_b} - \frac{E_x^2}{\Gamma_b^3} = 0$$

$$H_0 = \frac{1}{2} \vec{x}_1'^2 + \frac{E_{B0}^2}{2} \vec{x}_1^2 + \frac{e \phi_0}{m \delta b^3 \beta_b^2 c^2}$$



and assume small-amplitude axisymmetric ($\partial/\partial\theta=0$) perturbations with normal mode form:

$$\delta f(\vec{x}_1, \vec{x}_1', s) = \delta f(r, \vec{x}_1', k) e^{-iks}$$

$$\delta \phi(\vec{x}_1, s) = \delta \phi(r, k) e^{-iks}$$

$k = \text{const}$ (mode eigenfrequency)

The equilibrium characteristics in the core of the KV beam can be expressed as:

$$r^2(\tilde{s}) = r^2 \cos^2 [k_p(\tilde{s}-s)] + \frac{rr' \cos \psi \sin [2k_p(\tilde{s}-s)]}{k_p} + \frac{r'^2}{k_p^2} \sin^2 [k_p(\tilde{s}-s)]$$

$$x(\tilde{s}=s) = r \cos \theta \quad ; \quad x'(\tilde{s}=s) = r' \cos \theta_p$$

$$y(\tilde{s}=s) = r \sin \theta \quad ; \quad y'(\tilde{s}=s) = r' \sin \theta_p$$

$$\psi \equiv \theta - \theta_p$$

$$k_p = \left(k_{p0}^2 - \frac{Q}{r_b^2} \right)^{1/2} = \frac{\epsilon \lambda}{r_b^2} \quad \text{Depressed B-tran wavenumber of particle oscillations}$$

These results can be inserted into the characteristic equation

$$\delta f(\vec{x}_1, \vec{x}'_1, s) = \frac{g}{m \epsilon_0^3 \beta_b^2 c^2} \int_{-\infty}^s d\tilde{s} \frac{\partial}{\partial \vec{x}_1(\tilde{s})} \frac{\partial}{\partial \vec{x}'_1(\tilde{s})} f_0(H_0(\vec{x}_1(\tilde{s}), \vec{x}'_1(\tilde{s})))$$

to derive an expression for $\delta f(r, \vec{x}')$. This expression can then be inserted into the Poisson equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \delta \phi(r)}{\partial r} \right) = - \frac{g}{\epsilon_0} \int d^3 x' \delta f(r, \vec{x}')$$

to derive a linear eigenvalue equation for $\delta \phi(r)$:

A significant amount of manipulation obtains the following form for the eigenvalue equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \delta\phi(r) = \frac{\hat{\omega}_p^2}{\gamma_b \beta_b c^2} \mathcal{O}(r_b - r) \frac{1}{r'_1} \frac{\partial}{\partial r'_1} I_{\text{orb}}(r, r'_1, k) \quad (1)$$

$$r'_1 = \frac{E_x^2}{\gamma_b^2} \left(1 - \frac{r^2}{r_b^2} \right)$$

$$+ \frac{\hat{\omega}_p^2 / (\gamma_b \beta_b c^2)}{E_x^2 / \gamma_b^2} \delta(r - r_b) \left[\delta\phi + I_{\text{orb}}(r, r'_1, k) \right] \quad (2)$$

$$\cdot r'_1 = 0$$

Subject to: $\delta\phi(r=r_b) = 0$, $\hat{\omega}_p^2 = \frac{e^2 n}{\epsilon_{\text{eom}}} = \text{Plasma Freq. Squared.}$
where:

$$\mathcal{O}(r_b - r) = \begin{cases} 1 & r_b > r \\ 0 & r_b < r \end{cases} \quad \text{Heaviside Step Function}$$

$$I_{\text{orb}}(r, r'_1, k) = ik \int_{-\pi}^{\pi} \frac{d\Psi}{2\pi} \int_{-\infty}^s d\tilde{s} \delta\phi(r(\tilde{s}), k) e^{-ik(\tilde{s}-s)}$$

Orbit integral.

Note:

- Term (1) of $\mathcal{O}(r_b - r)$ is a body-wave perturbation existing only in the core ($r < r_b$) of the equilibrium beam.
- Term (2) of $\delta(r - r_b)$ is a surface-wave perturbation existing only at the edge ($r = r_b$) of the equilibrium beam.
- The orbit integral $I_{\text{orb}}(r, r'_1, k)$ depends on both $\delta\phi$ and the eigenfrequency k .

The Poisson equation has become a linear integro-differential eigenvalue equation fixing the mode perturbed potential $\delta\phi$ and the eigenfrequency k .

Glückstern Mode Solution S.M. Lund 4/

This eigenvalue equation is difficult, but it has been solved analytically.

- A finite polynomial in r^2 expansion of $\delta\phi$ for $r \leq r_b$ can satisfy the equation (terms truncate)
- Expansions are inserted into the characteristic integrals and coefficients are identified power-by-power in r^2 , and assembled.

Solution (after much analysis)

Eigenfunction:

$$\delta\phi_n(r) = \begin{cases} \frac{A_n}{2} [P_{n-1}(1 - 2r^2/r_b^2) + P_n(1 - 2r^2/r_b^2)], & 0 \leq r \leq r_b \\ 0, & r_b < r \leq r_p \end{cases}$$

$n = 1, 2, 3, \dots$

radial mode index

$A_n = \text{const.}$

linear mode amplitude.

$P_n(x)$

$n^{\text{'}}\text{th}$ order Legendre Polynomial

Dispersion Relation:

Each n -labeled eigenfunction has ω_n (degenerate)
 "eigenfrequencies" satisfying an n^{th} degree
 polynomial in k^2 dispersion relation.

$$\omega_n + \frac{1 - (\delta/\delta_0)^2}{(\delta/\delta_0)^2} \left[B_{n-1} \left(\frac{k^2/k_{p0}^2}{\delta/\delta_0} \right) - B_n \left(\frac{k^2/k_{p0}^2}{\delta/\delta_0} \right) \right] = 0$$

where: $\frac{\delta}{\delta_0} \equiv \frac{k_p}{k_{p0}} = \frac{(k_{p0}^2 - Q/r_b^2)^{1/2}}{k_{p0}}$

and

$$B_n(\alpha) \equiv \begin{cases} \frac{1}{[(\alpha/2)^2 - 0^2]} \cdot \frac{[(\alpha/2)^2 - 1^2]}{[(\alpha/2)^2 - 1^2]} \cdot \frac{[(\alpha/2)^2 - 2^2]}{[(\alpha/2)^2 - 2^2]} \cdots \frac{[(\alpha/2)^2 - (n-1)^2]}{[(\alpha/2)^2 - n^2]} & n=0 \\ \frac{[(\alpha/2)^2 - 1^2]}{[(\alpha/2)^2 - 1^2]} \cdot \frac{[(\alpha/2)^2 - 3^2]}{[(\alpha/2)^2 - 3^2]} \cdots \frac{[(\alpha/2)^2 - (n-1)^2]}{[(\alpha/2)^2 - n^2]} & n=1, 3, 5, \dots \\ \frac{[(\alpha/2)^2 - 2^2]}{[(\alpha/2)^2 - 2^2]} \cdot \frac{[(\alpha/2)^2 - 4^2]}{[(\alpha/2)^2 - 4^2]} \cdots \frac{[(\alpha/2)^2 - (n-1)^2]}{[(\alpha/2)^2 - n^2]} & n=2, 4, 6, \dots \end{cases}$$

Properties:

Radial Eigenfunction:

- Vanishes outside the equilibrium beam edge ($r > r_b$).
- Has $n-1$ nodes with $\delta\phi = 0$ within the equilibrium beam ($r < r_b$).
- Each n labeled eigenfunction has z_n distinct frequencies.

Corresponding perturbed density can be calculated from Poisson's equation:

$$\boxed{\delta n_n = \delta n_n(r) e^{-i\omega t}}$$

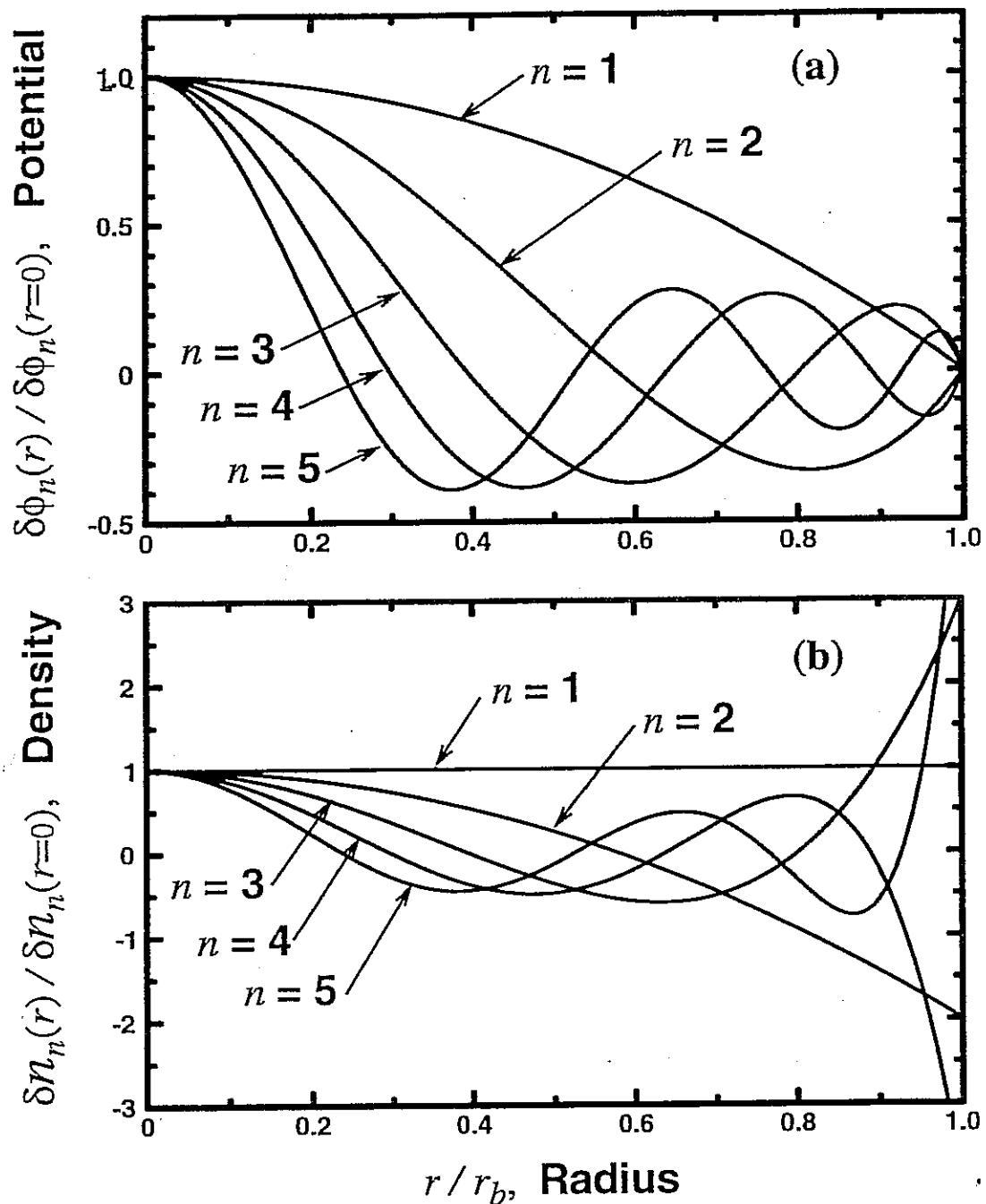
$$\boxed{\delta n_n(r) = -\frac{e_0 k}{2} \frac{\partial}{\partial r} \left(r^2 \frac{\delta \phi_n}{\delta r} \right)}$$

- Find that the perturbed density of the mode is more larger near the outer ($r \approx r_b$) edge of the beam for larger n .

Eigenfunction Form:

Mode number n	$\delta\phi_n/A_n$ (potential)	δn_n (density, scaled units)
1	$1 - \tilde{r}^2$	1
2	$1 - 4\tilde{r}^2 + 3\tilde{r}^4$	$4(1 - 3\tilde{r}^2)$
3	$1 - 9\tilde{r}^2 + 18\tilde{r}^4 - 10\tilde{r}^6$	$9(1 - 8\tilde{r}^2 + 10\tilde{r}^4)$
4	$1 - 16\tilde{r}^2 + 60\tilde{r}^4 - 80\tilde{r}^6 + 35\tilde{r}^8$	$16(1 - 15\tilde{r}^2 + 45\tilde{r}^4 - 35\tilde{r}^6)$
⋮	⋮	⋮

$\tilde{r} \equiv r/r_b$

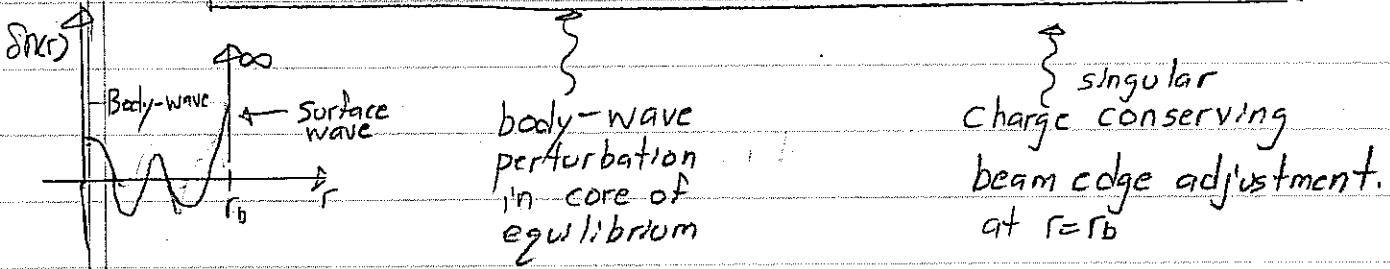
Radial Eigenfunction

Perturbations should introduce no net charge into the system:

$$2\pi \int_0^{\Gamma_b} dr r \delta n(r) = 0$$

The $r < \Gamma_b$ component of the perturbations are not the only terms present. For the $r < \Gamma_b$ eigenfunctions calculated $\int_0^{\Gamma_b} dr r \delta n(r) \neq 0$. A more detailed analysis shows that:

$$\delta n_n(r) = \delta n_n(r) \Big|_{\text{body}} \delta(\Gamma_b - r) + \delta n_n \Big|_{\text{surface}} \frac{\Gamma_b^2}{r} \delta(r - \Gamma_b)$$



where:

$$\delta n_n \Big|_{\text{body}} = -\frac{\epsilon_0}{2} \frac{1}{\Gamma} \frac{\partial}{\partial r} \left(r \frac{\partial \delta n}{\partial r} \right)$$

$$\delta n_n \Big|_{\text{surface}} = \text{const} \times (-1)^n n A_n$$

To linear order this is equivalent to:

$$n(r) = [n + \delta n_n(r)] \Big|_{\text{body}} \delta [\Gamma_b + \delta \Gamma_b - r]$$

$$\delta \Gamma_b = \text{const} \times (-1)^n n A_n$$

readjustment of
beam edge radius.

Dispersion Relation

- Polynomial in $k^2 \Rightarrow \pm k$ solutions, and therefore there will be unstable growing perturbations if k is complex:

$$\delta\phi \sim \delta\phi_n(r) e^{-ikr}$$

$$k = k_r \pm ik_I \quad k_r = \text{real part}$$

$$k_I = \text{imaginary part}$$

For the unstable branch:

$$\delta\phi \sim \delta\phi_n(r) e^{-ik_r r} \cdot e^{ik_I r} \Rightarrow \text{exponential growth.}$$

- $|k|$ is a function of n and σ/σ_0 only.

— $0 \leq \sigma/\sigma_0 \leq 1$

\uparrow
strongest
possible space
charge.

\uparrow
zero space-charge,

- Instabilities will occur over a range of σ/σ_0 and will turn off for σ/σ_0

large enough (weak space-charge).
KV beam is always stable for zero space-charge since orbits are stable.

Dispersion Relations:

Mode number n

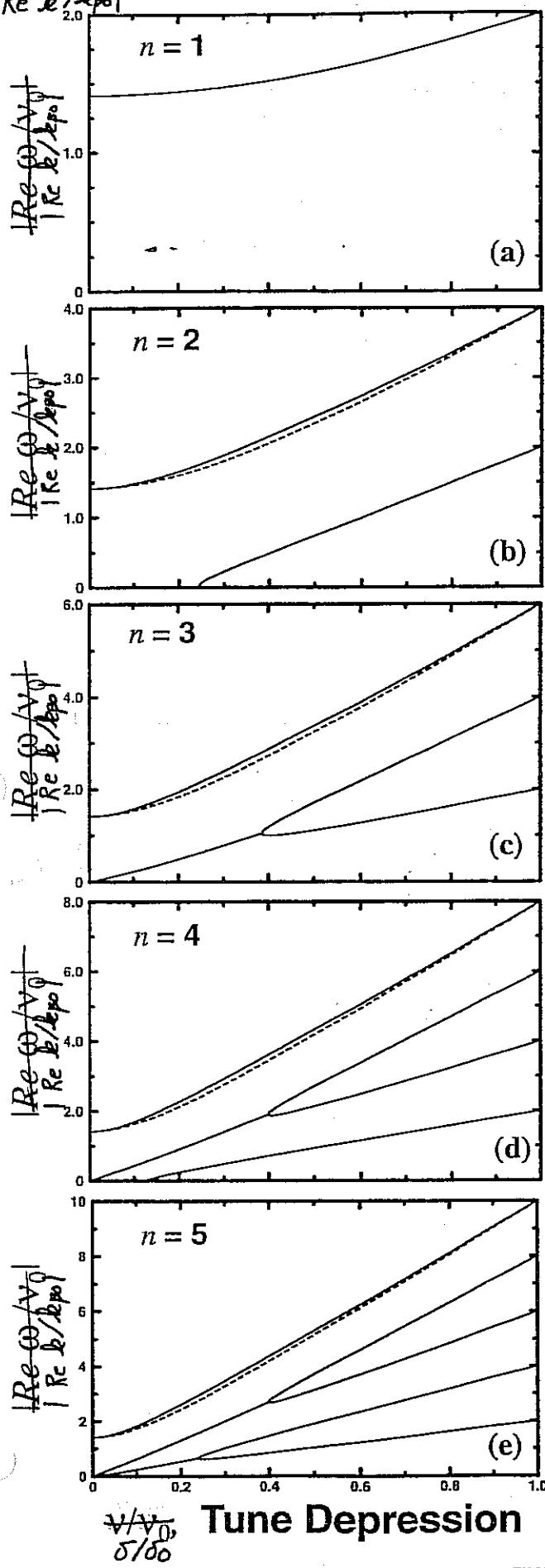
Dispersion relation

$$1 \quad (k/k_{p0})^2 - 2(1 + \sigma^2/\sigma_0^2) = 0$$

$$2 \quad (k/k_{p0})^4 - 2(1 + 9\sigma^2/\sigma_0^2)(k/k_{p0})^2 - 4(\sigma^2/\sigma_0^2)(1 - 17\sigma^2/\sigma_0^2) = 0$$

Rapidly more complicated!

$|\text{Re } \omega|/\sqrt{\nu_0}$, Oscillation Frequency



$|\text{Im } \omega|/\sqrt{\nu_0}$, Growth Rate

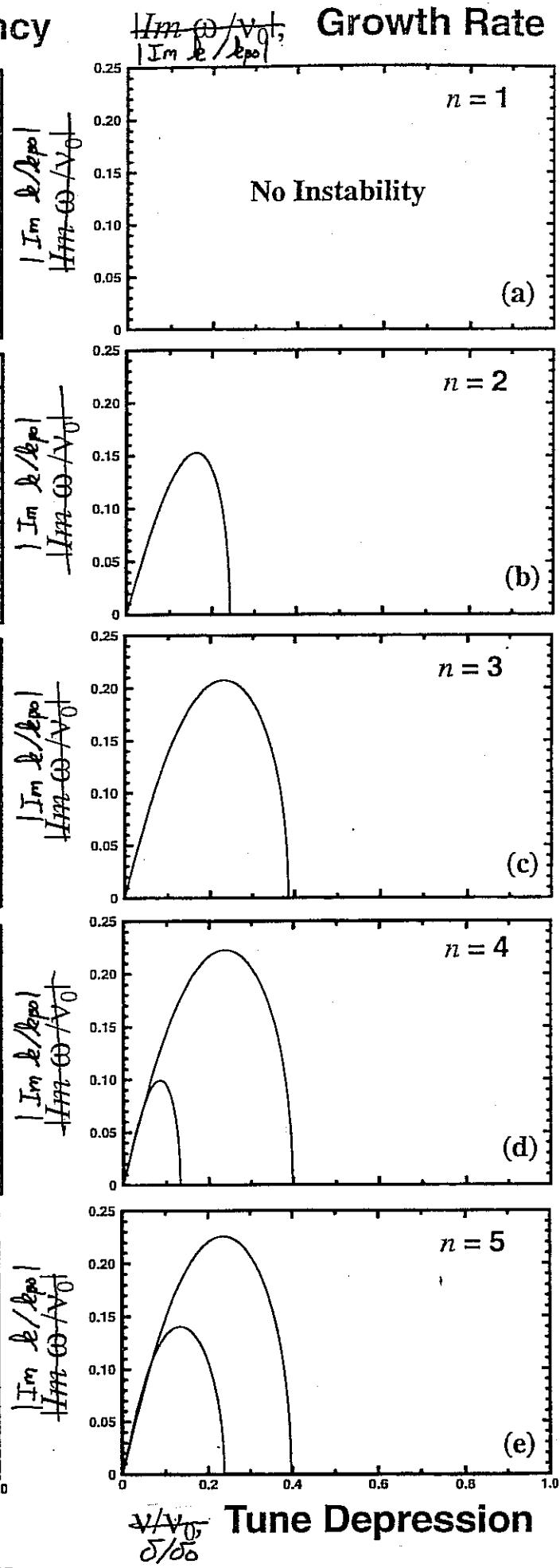


Fig. 8

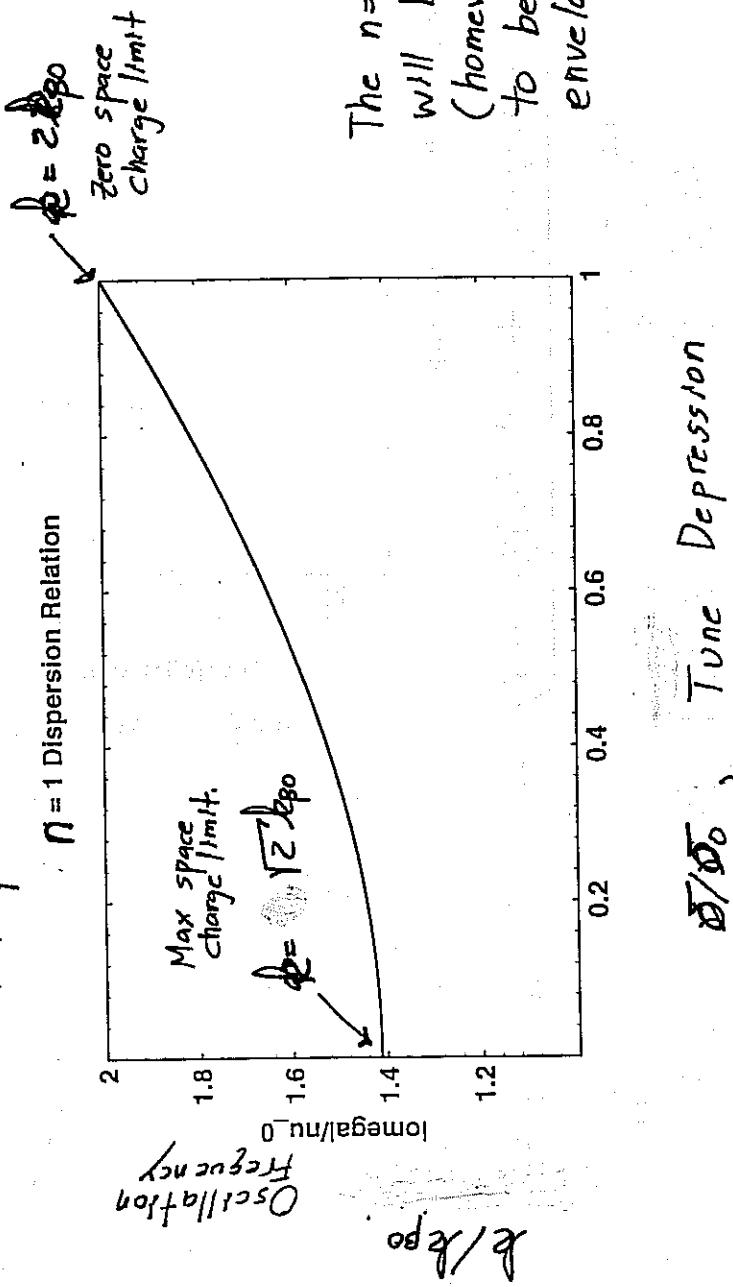
9

Kinetic Theory – Transverse Gluckstern Modes (6)

Example: $n = 1$, Envelope Mode

$$\delta\phi_1 = \begin{cases} A_1 [1 - (r/r_b)^2], & 0 \leq r \leq r_b, \\ 0, & r_b \leq r \leq r_p, \end{cases}$$

$$(\omega/\omega_0)^2 = 2 + 2(\delta/\delta_0)^2$$



Oscillation Frequency

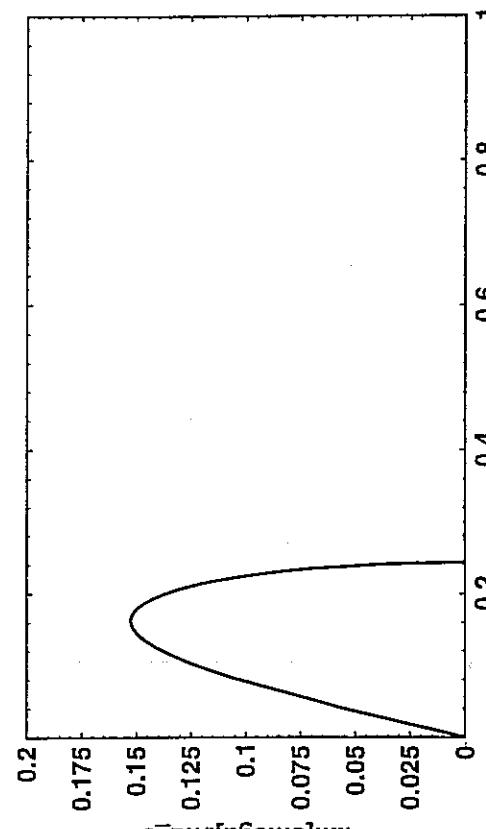
Kinetic Theory – Transverse Gluckstern Modes (7)

Example: $n = 2$ Mode

$$\delta\phi_2 = \begin{cases} A_2[1 - 4(r/r_b)^2 + 3(r/r_b)^4], & 0 \leq r \leq r_b, \\ 0, & r_b \leq r \leq r_p, \end{cases}$$

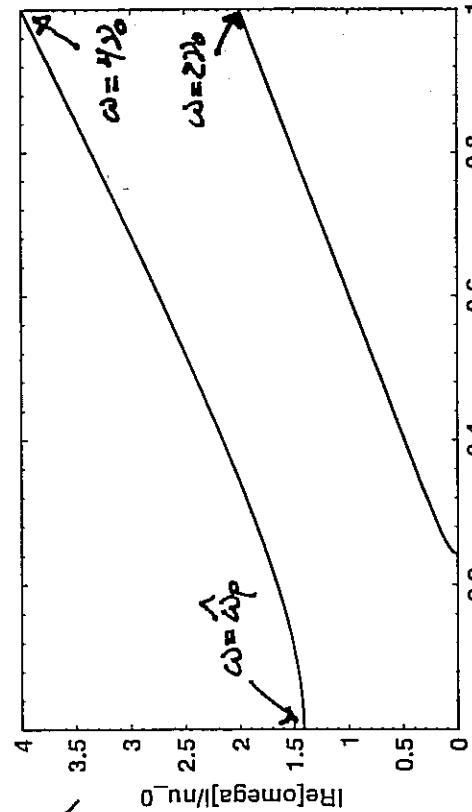
$$(\omega/\nu_0)^2 = 1 + 9(\nu/\nu_0)^2 \pm \sqrt{1 + 22(\nu/\nu_0)^2 + 13(\nu/\nu_0)^4}$$

$j = 2$ Dispersion Relation -- Imaginary



$|Im[\omega]/\nu_0|$, Growth Rate

$j = 2$ Dispersion Relation -- Real



ω/ν_0 , Tuning Depression

ω/ν_0 , Tuning Depression

Oscillation Frequency

Growth Rate

As might be expected on physical grounds, the singular KV distribution drives numerous, strong, collective instabilities. This implies that the KV model is suspect since real beams are often transported where the KV model would predict strong instability. However:

- Low-order KV features (envelope modes) are correct and well verified.
- Higher order collective modes observed on intense beam cores often look similar to the KV model predictions in density/potential etc, but are not unstable.

How is this situation resolved? A partial answer was suggested by a fluid model developed by Lund and Davidson. In this model:

- Density and temperature profiles (i.e., low order features) of the KV model were preserved.
 - The singular phase-space structures were eliminated.
- A stability analysis obtained:

$$\text{Mode Eigenfunction: } \delta\phi_n = \frac{A_n}{Z} \left(P_{n-1}(1 - Z \frac{r^2}{r_b^2}) + P_n(1 - Z \frac{r^2}{r_b^2}) \right)$$

$$\text{Mode Dispersion Relation: } \left(\frac{k}{k_{B0}} \right)^2 = Z + Z \left(\frac{\sigma}{\sigma_0} \right)^2 (2n^2 - 1)$$

$$n = 1, 2, 3, \dots$$

Features of Fluid model:

S.M. Lund 13/

- Identical radial eigenfunction to the full kinetic theory
- Fluid mode dispersion relation predicts stability for all modes and closely tracks the (stable) high frequency branch of the KV dispersion relation for the full range of space charge strength $0 \leq \delta/\delta_0 \leq 1$
 - Fluid mode dispersion relation plotted dashed on KV mode plots.
 - The $n=1$ fluid envelope mode is identical to the KV envelope mode.

Since the fluid model reproduces the coarse-macroscopic features of the KV model - which can be a good approximation at high space-charge intensities, this implies:

- KV-model mode eigenfunctions should roughly model those of intense beams with smooth distributions.
- Oscillation frequencies may be close to the (stable) high frequency KV mode branch
 - May be other lower frequency branches that are also physical.
- Many high-order KV instabilities may be of little relevance to real beams.
 - Low order (envelope and maybe others) can be relevant.

The real issue for high intensity collective modes may not be higher order KV instabilities but if low-order collective modes can:

- Be driven unstable by periodic (s-varying) focusing structures in machine lattices, errors in rings, etc.
- Drive the production of beam halo, etc.

References:

Material on the kinetic stability of KV beams is found mostly in journals.

Original references

Gluckstern, Proc. 1970 Proton Linac Conference, Natl. Accel. Lab., pg. 811 — First KV mode analysis.

T.F. Wang and L. Smith, Part. Accel. 12, 247 (1982). — Simplified (closed form) mode eigenfunction and dispersion relation.

Interpretation of Branches, Mode Structure, KV Fluid Stability, S.M. Lund and R.C. Davidson, Physics of Plasmas 5, 3028 (1998). Detailed analysis of eigenfunctions, dispersion relations, etc. in appendices. Fluid mode analysis and interpretations of KV modes.

Other papers by Hoffmann, Gluckstern, and others. Hoffmann et al. analyzed KV in periodic focusing lattices.